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Abstract

We study a family of non-linear McKean-Vlasov SDEs driven by a Poisson measure, modelling the mean-field asymptotic of a network of generalized Integrate-and-Fire neurons. We give sufficient conditions to have periodic solutions through a Hopf bifurcation. Our spectral conditions involve the location of the roots of an explicit holomorphic function. The proof relies on two main ingredients. First, we introduce a discrete time Markov Chain modeling the phases of the successive spikes of a neuron. The invariant measure of this Markov Chain is related to the shape of the periodic solutions. Secondly, we use the Lyapunov-Schmidt method to obtain self-consistent oscillations. We illustrate the result with a toy model for which all the spectral conditions can be analytically checked.
Keywords McKean-Vlasov SDE · Long time behavior · Hopf bifurcation · Mean-field interaction · Volterra integral equation · Piecewise deterministic Markov process
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Contents

1 Introduction

We consider a mean-field model of spiking neurons. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$, $b : \mathbb{R}_+ \to \mathbb{R}$ such that $b(0) \geq 0$. Let $\mathbf{N}(du, dz)$ be a Poisson measure on $\mathbb{R}^2_+$ with intensity the Lebesgue measure $dudz$. Consider the following McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t b(X_u)du + J \int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-} \mathbb{1}_{\{z \leq f(X_{u-})\}} \mathbf{N}(du, dz). \tag{1}$$

Here, $J \geq 0$ is a deterministic constant (it models the strength of the interactions) and the initial condition X_0 is independent of the Poisson measure. Informally, the SDE (1) can be understood in the following sense: Between the jumps, X_t solves the scalar ODE $\dot{X}_t = b(X_t) + J\, \mathbb{E}\, f(X_t)$ and X_t jumps to 0 at rate $f(X_t)$.

This SDE is non-linear in the sense of McKean-Vlasov, because of the interaction term $\mathbb{E}\, f(X_t)$ which depends on the law of X_t. Let $\nu(t, dx) := \mathcal{L}(X_t)$ be the law of X_t. It solves the following non-linear Fokker-Planck equation, in the sense of measures:

$$\begin{aligned} &\partial_t \nu(t, dx) + \partial_x \left[(b(x) + J r_t)\nu(t, dx)\right] + f(x)\nu(t, dx) = r_t \delta_0 \\ &\nu(0, dx) = \mathcal{L}(X_0), \quad r_t = \int_{\mathbb{R}_+} f(x)\nu(t, dx). \end{aligned} \tag{2}$$

Here δ_0 is the Dirac measure in 0. If furthermore $\mathcal{L}(X_t)$ has a density for all t, that is $\mathcal{L}(X_t) = \nu(t, x)dx$ then $\nu(t, x)$ solves the following strong form of (2)

$$\begin{aligned} &\partial_t \nu(t, x) + \partial_x \left[(b(x) + J r_t)\nu(t, x)\right] + f(x)\nu(t, x) = 0, \\ &\nu(0, x)dx = \mathcal{L}(X_0), \quad r_t = \int_{\mathbb{R}_+} f(x)\nu(t, x)dx, \end{aligned}$$

with the boundary condition

$$\forall t > 0, \quad (b(0) + J r_t)\, \nu(t, 0) = r_t.$$

We study the existence of periodic solution to this non-linear Fokker-Planck equation. We give sufficient conditions for the existence of a Hopf bifurcation around a stationary solution of (2).

Associated particle system

Equations (1) and (2) appeared (see e.g. [DGLP15]) as the limit of the following networks of neurons. For each $N \geq 1$, consider *i.i.d.* initial conditions $(X_0^{i,N})_{i \in \{1,..,N\}}$ with law $\mathcal{L}(X_0)$. The *càdlàg* process $(X_t^{i,N})_{i \in \{1,..,N\}} \in \mathbb{R}^N$ is a PDMP: between the jumps each $X_t^{i,N}$ solves the ODE $\dot{X}_t^{i,N} = b(X_t^{i,N})$ and "spikes" with rate $f(X_t^{i,N})$. When a spike occurs, say neuron i spikes at (random) time τ, its potential is reset to 0 while the others receive a "kick" of size $\frac{J}{N}$:

$$X_{\tau_+}^{i,N} = 0, \quad \text{and} \quad \forall j \neq i, \quad X_{\tau_+}^{j,N} = X_{\tau_-}^{j,N} + \frac{J}{N}.$$

This completely defines the particle systems. Note that the parameter J models the size of the interactions between two neurons. As N goes to infinity, a phenomena of *propagation of chaos* occurs. Each neuron, say $(X_t^{1,N})_{t \geq 0}$, converges in law to the solution of (1). We refer to [FL16] for a proof of such convergence result under stronger assumptions. There is a qualitative difference between the particle systems and the solution of the limit equation (1): for a fixed value of N, the particle system is Harris ergodic (see [DO16], where this result is proved under stronger assumptions on b and f) and so it admits a unique, globally attractive, invariant measure. In particular, there are no stable oscillations when the number of particles is finite. For the limit equation however, the long time behavior is richer: for fixed values of the parameters there can be multiple invariant measures (see [CTV20] and [Cor20] for some explicit examples) and, as shown here, there can exist periodic solutions.

Literature

From a mathematical point of view, this model has been first introduced by [DGLP15], after many considerations by physicists (see for instance [PG00], [GKNP14] and [Ces11] and references therein). Study of existence and path-wise uniqueness of (1), convergence of the particle system are addressed in [FL16]. The long time behavior of (1) is studied in [CTV20] in the case of weak interactions: b and f being fixed, the authors prove that there exists a constant $\bar{J}$ (depending on b and f) such that for all $J < \bar{J}$, (1) admits a unique globally attractive invariant measure. Finally in [Cor20], the local stability of an invariant measure is studied with no further assumptions on the size of the interactions J. It is proved that the stability of an invariant measure is given by the location of the roots of some holomorphic function. In [LM20], the authors study a "metastable" behavior of the particle system. They give examples of drifts b and rate functions f where the particle system follows the long time behavior of the mean-field model for an exponential large time, before finally converging to its (unique) invariant probability measure.

This model belongs to the class of generalized integrate-and-fire neurons, whose most celebrated example is the "fixed threshold" model (see for instance [CCP11], [DIRT15] and the references therein). Many of the techniques developed here also apply to this variant.

In [DV17], numerical evidences are given of the existence of a Hopf bifurcation in a close setting: the dynamics between the jumps is (as in [DGLP15]) given by

$$\dot{X}_t = -(X_t - \mathbb{E}\, X_t) + J\, \mathbb{E}\, f(X_t).$$

In particular the potentials of each neuron are attracted to their common mean. This models "electrical synapses", while $J\,\mathbb{E}\,f(X_t)$ models the chemical synapses. Oscillations with both electrical and chemical synapses is also studied in a different model in [PDRDM19]. In this work, the mean-field equation is a 2D-ODE and so the analysis of the Hopf bifurcation is standard. Finally, oscillations with multi-populations, in particular with both excitatory and inhibitory neurons have been extensively studied in neuroscience. For instance in [DL17], it is shown that multi-populations of mean-field Hawkes processes can oscillate. Again, the dynamics is reduced to a finite dimension ODE.

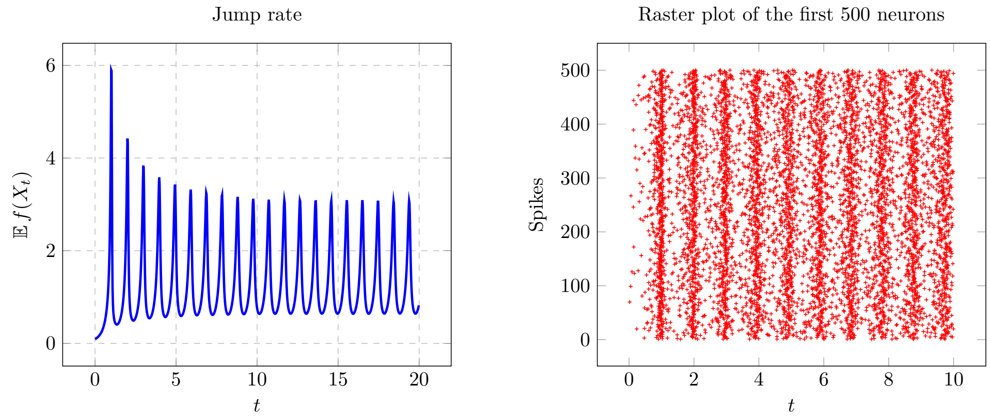
It is well-known that the long time behavior of McKean-Vlasov SDEs can be significantly different from markovian SDEs. In [Sch85b] and [Sch85a], the author give simple examples of such non-linear SDEs which oscillate. Again, in these examples, the dynamic can be reduced to an ordinary differential equation. To go beyond ODEs, the framework of Delay differential equation is often used: see for instance [Sta87] for the study of Hopf bifurcations for such equations, based on the Lyapunov-Schmidt method. In [LP20a; LP20b] the authors study periodic solutions of a McKean-Vlasov SDE using a slow-fast approach. Another approach is to use the center manifold theory to reduce infinite dimensional problem to manifold of finite dimension: we refer to [HI11] (see also [GPPP12] for an application to some McKean-Vlasov SDE). Finally, in [Kie12] an abstract framework is presented to study Hopf bifurcations for some classes of regular PDEs. Even though our proof is not based on the PDE (2) (but on the Volterra integral equation described below), we follow the methodology of [Kie12] to obtain our main result.

The Volterra integral equation

As in [CTV20; Cor20], we study the long time behavior of the solution of (1) through its "linearized" version: given a non-negative scalar function $\boldsymbol{a} \in L^\infty(\mathbb{R}_+;\mathbb{R}_+)$, consider the non-homogeneous linear SDE:

$$\forall t \geq s, \quad Y^{\boldsymbol{a},\nu}_{t,s} = Y_s + \int_s^t [b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u] du - \int_s^t \int_{\mathbb{R}_+} Y^{\boldsymbol{a},\nu}_{u-,s} \mathbb{1}_{\{z \leq f(Y^{\boldsymbol{a},\nu}_{u-,s})\}} \mathbf{N}(du, dz), \tag{3}$$

starting with law ν at time s. That is, equation (3) is (1) where the interactions $J\,\mathbb{E}\,f(X_u)$ have been replaced by the "external current" a_u. For all $t \geq s$ and for all $\boldsymbol{a} \in L^\infty(\mathbb{R}_+;\mathbb{R}_+)$, consider



(a) (b)

Figure 1: Consider the following example where for all $x \geq 0$, $f(x) = x^{10}$, $b(x) = 2 - 2x$ and $J = 0.8$. Using a Monte-Carlo method, we simulate the particle systems with $N = 8 \cdot 10^5$ neurons, starting at $t = 0$ with i.i.d. uniformly distributed random variables on $[0, 1]$. Stable oscillations appear. (a) Empirical mean number of spikes per unit of time. (b) Each red crosses corresponds to a spike of one of the first 500 neurons (spike raster plot).

$\tau_s^{\boldsymbol{a},\nu}$ the time of the first jump of $Y^{\boldsymbol{a},\nu}$ after s

$$\tau_s^{\boldsymbol{a},\nu} := \inf\{t \geq s : Y_{t,s}^{\boldsymbol{a},\nu} \neq Y_{t-,s}^{\boldsymbol{a},\nu}\}. \tag{4}$$

We introduce the spiking rate $r_{\boldsymbol{a}}^\nu(t, s)$, the survival function $H_{\boldsymbol{a}}^\nu(t, s)$ and the density of the first jump $K_{\boldsymbol{a}}^\nu(t, s)$ to be

$$r_{\boldsymbol{a}}^\nu(t, s) := \mathbb{E}\, f(Y_{t,s}^{\boldsymbol{a},\nu}), \quad H_{\boldsymbol{a}}^\nu(t, s) := \mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t), \quad K_{\boldsymbol{a}}^\nu(t, s) := -\frac{d}{dt}\mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t). \tag{5}$$

Notation 1. *To simplify the notations, when $\nu = \delta_x$, we write*

$$r_{\boldsymbol{a}}^x(t, s) := r_{\boldsymbol{a}}^{\delta_x}(t, s).$$

When $\nu = \delta_0$ (that is when $x = 0$), we may remove the x superscript and write

$$r_{\boldsymbol{a}}(t, s) := r_{\boldsymbol{a}}^{\delta_0}(t, s).$$

Finally, when $\boldsymbol{a}$ is constant and equal to $\alpha \geq 0$, it holds that $r_\alpha^\nu(t, s) = r_\alpha^\nu(t - s, 0)$ and we simply note $r_\alpha^\nu(t) := r_\alpha^\nu(t, 0)$. Finally, we extend the three functions for $s > t$ by setting

$$\forall s > t, \quad r_{\boldsymbol{a}}^\nu(t, s) := 0.$$

We use the same conventions for H and K.

It is known from [CTV20, Prop. 19] (see also [Cor20, Prop. 6] for a shorter proof) that $r_{\boldsymbol{a}}^\nu$ is the solution of the following Volterra integral equation

$$r_{\boldsymbol{a}}^\nu(t, s) = K_{\boldsymbol{a}}^\nu(t, s) + \int_s^t K_{\boldsymbol{a}}(t, u) r_{\boldsymbol{a}}^\nu(u, s) du. \tag{6}$$

Moreover, by [CTV20, Lem. 17], one has

$$1 = H_{\boldsymbol{a}}^\nu(t, s) + \int_s^t H_{\boldsymbol{a}}(t, u) r_{\boldsymbol{a}}^\nu(u, s) du. \tag{7}$$

Following [GLS90], given $a, b : \mathbb{R}^2 \to \mathbb{R}$ two measurable functions, it is convenient to use the notation

$$(a * b)(t, s) = \int_s^t a(t, u)b(u, s)du,$$

such that (6) and (7) simply write

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}} \quad \text{and} \quad 1 = H^\nu_{\boldsymbol{a}} + H_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}}.$$

The invariant measures of (1)**.**

Let $\alpha > 0$, define

$$\sigma_\alpha := \inf\{x \geq 0,\ b(x) + \alpha = 0\}, \tag{8}$$

and

$$\nu^\infty_\alpha(x) := \frac{\gamma(\alpha)}{b(x) + \alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y) + \alpha} dy\right) \mathbb{1}_{[0, \sigma_\alpha)}(x), \tag{9}$$

where $\gamma(\alpha)$ is the normalizing factor, such that $\int_{\mathbb{R}_+} \nu^\infty_\alpha(x)dx = 1$. Note that $\gamma(\alpha)$ is the jump rate under ν^∞_α:

$$\gamma(\alpha) = \nu^\infty_\alpha(f).$$

By [CTV20, Prop. 26], ν^∞_α is the unique invariant measure of the linear SDE (3) driven by the constant "external current" $\boldsymbol{a} \equiv \alpha$. Define

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)}. \tag{10}$$

It is readily seen that ν^∞_α is an invariant measure of the non-linear equation (1) with $J = J(\alpha)$. Reciprocally, for a fixed value of J, the number of invariant measures of (1) is the number of solutions $\alpha \geq 0$ to the scalar equation

$$\alpha = J\gamma(\alpha). \tag{11}$$

Any such invariant measure is characterized by its corresponding value of α.

Stability of an invariant measure

Let $\nu^\infty_{\alpha_0}$ be an invariant measure of (1), for some $\alpha_0 > 0$ satisfying (11). A sufficient condition for $\nu^\infty_{\alpha_0}$ to be locally stable is given in [Cor20]. First, consider $H_{\alpha_0}(t)$, defined by (5) (with $\nu = \delta_0$ and $\boldsymbol{a} \equiv \alpha_0$). For $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary part. The Laplace transform of $H_{\alpha_0}(t)$ is defined for z with $\Re(z) > -f(\sigma_{\alpha_0})$ (σ_{α_0} is given by (8)):

$$\widehat{H}_{\alpha_0}(z) := \int_0^\infty e^{-zt} H_{\alpha_0}(t)dt.$$

Let

$$\lambda^*_{\alpha_0} := -\sup\{\Re(z) |\ \Re(z) > -f(\sigma_{\alpha_0}),\ \widehat{H}_{\alpha_0}(z) = 0\}. \tag{12}$$

The constant $\lambda^*_{\alpha_0}$ is related to the rate of convergence of $(Y^{\alpha,\nu}_{t,0})$ to its invariant measure $\nu^\infty_{\alpha_0}$. In particular we have

$$\forall \lambda < \lambda^*_{\alpha_0}, \quad \sup_{t \geq 0} |r^\nu_{\alpha_0}(t) - \gamma(\alpha_0)| e^{\lambda t} < \infty. \tag{13}$$

This describes the long time behavior of an isolated neuron subject to a constant current $\alpha_0 > 0$. Following [Cor20], we define for all $t \geq 0$

$$\Theta_{\alpha_0}(t) := \int_0^\infty \frac{d}{dx} r^x_{\alpha_0}(t) \nu^\infty_{\alpha_0}(dx). \tag{14}$$

Assume that

Assumption 2. *The drift* $b : \mathbb{R}_+ \to \mathbb{R}$ *is* $\mathcal{C}^2$*, with* $b(0) \geq 0$ *and*

$$\sup_{x \geq 0} |b'(x)| + |b''(x)| < \infty.$$

Assumption 3. *Consider* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

3.1 the function f *belongs to* $\mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$*,* $f(0) = 0$ *and* f *is strictly increasing on* $\mathbb{R}_+$*.*

3.2 one has $\sup_{x \geq 1} [f'(x)/f(x) + |f''(x)|/f(x)] < \infty$*.*

3.3 for all $A \geq 0$*,*

$$\sup_{x \geq 0} A f'(x)\,(1 + b(x)) - f^2(x) < \infty$$

and

$$\sup_{x \geq 0} A f'(x) - f(x) < \infty.$$

3.4 the function f *grows at most at a polynomial rate: there exists* $p > 0$ *such that* $\sup_{x \geq 1} \frac{f(x)}{x^p} < \infty$*.*

3.5 there exists a constant C *such that for all* $x, y \geq 0$*,*

$$f(xy) \leq C(1 + f(x))(1 + f(y)).$$

Assumption 4. *Assume moreover that the deterministic flow is not degenerate at* σ_{α_0} *in the following sense*

$$\sigma_{\alpha_0} < \infty \quad \textit{and} \quad b'(\sigma_{\alpha_0}) < 0 \tag{15}$$

$$\textbf{\textit{or}} \qquad \sigma_{\alpha_0} = \infty \quad \textit{and} \quad \inf_{x \geq 0} b(x) + \alpha_0 > 0. \tag{16}$$

Theorem 5 ([Cor20])**.** *Grant Assumptions 2, 3 and 4. It holds that* $\lambda^*_{\alpha_0} > 0$*, and for all* $\lambda < \lambda^*_{\alpha_0}$ *one has* $t \mapsto e^{\lambda t}\Theta_{\alpha_0}(t) \in L^1(\mathbb{R}_+)$*, so that* $z \mapsto \widehat{\Theta}_{\alpha_0}(z)$ *is holomorphic on* $\Re(z) > \lambda^*_{\alpha_0}$*. Assume that*

$$\sup\{\Re(z) \mid z \in \mathbb{C},\ \Re(z) > -\lambda^*_{\alpha_0},\ J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) = 1\} < 0, \tag{17}$$

then the invariant measure $\nu^\infty_{\alpha_0}$ *is locally stable.*

We refer to [Cor20, Def. 16] for definition of local stability, in particular for the definition of the distance between two probability measures.

Under Assumptions 2 and 3, the function $\alpha \mapsto J(\alpha)$ is $\mathcal{C}^2$ on $\mathbb{R}^*_+$. Assume J is small enough such that for some $\alpha > 0$ one has $J(\alpha)||\Theta_\alpha||_1 < 1$, and so the invariant measure ν^∞_α is locally stable. There are two "canonical" ways to break (17) at some *bifurcation point* α_0: either there exists some $\tau_0 > 0$ such that $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\pm\frac{i}{\tau_0}) = 1$ or $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(0) = 1$. The first case is the subject of this paper: we make explicit sufficient condition to have a Hopf bifurcation.

In the second case, the following lemma shows that $J'(\alpha_0) = 0$. So, at least in the non-degenerate case where $J''(\alpha_0) \neq 0$, the function $\alpha \mapsto J(\alpha)$ is not strictly monotonic in the neighborhoods of α_0: this is a static bifurcation which typically leads to *bistability* (or *multistability*, etc.).

Lemma 6. *Under Assumptions 2 and 3, it holds that for all* $\alpha > 0$*,*

$$J'(\alpha) = \frac{1 - J(\alpha)\widehat{\Theta}_\alpha(0)}{\gamma(\alpha)}.$$

Proof. First, recall that $J(\alpha) = \frac{\alpha}{\gamma(\alpha)}$. So it suffices to show that $\gamma'(\alpha) = \widehat{\Theta}_\alpha(0)$. By [CTV20, eq. (31)], one has $\gamma(\alpha)^{-1} = \widehat{H}_\alpha(0)$. So we have to prove that

$$\frac{d}{d\alpha}\widehat{H}_\alpha(0) = -\frac{\widehat{\Theta}_\alpha(0)}{[\gamma(\alpha)]^2}.$$

Following [Cor20], let:

$$\Psi_\alpha(t) := -\int_0^{\sigma_\alpha} \frac{d}{dx} H_\alpha^x(t) \nu_\alpha^\infty(x) dx. \tag{18}$$

It holds that ([Cor20])

$$\Psi_\alpha(t) = \gamma(\alpha) \int_0^\infty H_\alpha(t+u) \frac{f(\varphi_{t+u}^\alpha(0)) - f(\varphi_u^\alpha(0))}{b(\varphi_u^\alpha(0)) + \alpha} du. \tag{19}$$

Moreover, let

$$\forall t \geq 0, \quad \Xi_\alpha(t) := \frac{d}{dt}\Psi_\alpha(t).$$

We have

$$\forall t \geq 0, \quad \Xi_\alpha(t) = \int_0^{\sigma_\alpha} \frac{d}{dx} K_\alpha^x(t) \nu_\alpha^\infty(x) dx. \tag{20}$$

So, using (6) (with $\nu = \delta_x$ and $\boldsymbol{a} \equiv \alpha$) we deduce that (see [Cor20, eq. (40)] for more details):

$$\Theta_\alpha = \Xi_\alpha + r_\alpha * \Xi_\alpha. \tag{21}$$

Note that $\Psi_\alpha(0) = 0$, $\lim_{t\to\infty} \Psi_\alpha(t) = 0$ and so $\widehat{\Xi}_\alpha(0) = 0$. Let

$$\xi_\alpha(t) := r_\alpha(t) - \gamma(\alpha).$$

Using (13) (with $\nu = \delta_0$), we deduce that $\xi_\alpha \in L^1(\mathbb{R}_+)$. So (21) yields

$$\Theta_\alpha = \Xi_\alpha + \gamma(\alpha)\Psi_\alpha + \xi_\alpha * \Xi_\alpha.$$

We deduce that $\widehat{\Theta}_\alpha(0) = \gamma(\alpha)\widehat{\Psi}_\alpha(0)$. Finally, we have

$$\begin{aligned}
\frac{d}{d\alpha}\widehat{H}_\alpha(0) &= \int_0^\infty \frac{d}{d\alpha} H_\alpha(t) dt \\
&= -\int_0^\infty H_\alpha(t) \int_0^t \frac{f(\varphi_t^\alpha(0)) - f(\varphi_\theta^\alpha(0))}{b(\varphi_\theta^\alpha(0)) + \alpha} d\theta dt \\
&= -\int_0^\infty \int_0^\infty H_\alpha(u+\theta) \frac{f(\varphi_{u+\theta}^\alpha(0)) - f(\varphi_\theta^\alpha(0))}{b(\varphi_\theta^\alpha(0)) + \alpha} d\theta du \quad \text{(using Fubini} \\
&\qquad \text{and the change of variables } u = t - \theta). \\
&= -\frac{\widehat{\Psi}_\alpha(0)}{\gamma(\alpha)}.
\end{aligned}$$

This ends the proof. ∎

The paper is structured as follows: in Section 2, we state the spectral assumptions and the main result, Theorem 14. We give a layout of its proof at the end of Section 2. In Section 3, we give the proof of Theorem 14. Finally, in Section 4, we give an explicit example of a drift b and a rate function f for which such Hopf bifurcations occur and the spectral assumptions can be analytically checked.

2 Assumptions and main result

Following [CTV20; Cor20], we assume that the law of the initial condition belongs to

$$\mathcal{M}(f^2) := \{\nu \in \mathcal{P}(\mathbb{R}_+) : \quad \int_{\mathbb{R}_+} f^2(x)\nu(dx) < \infty\}.$$

For such initial condition, under Assumptions 2 and 3, the non-linear SDE (1) has a unique path-wise solution (see [Cor20, Th. 9]).

Definition 7. *A family of probability measures $(\nu(t))_{t\in[0,T]}$ is said to be a T-periodic solution of* (1) *if*

1. $\nu(0) \in \mathcal{M}(f^2)$.
2. *For all $t \in [0, T]$, $\nu(t) = \mathcal{L}(X_t)$ where $(X_t)_{t\in[0,T]}$ is the solution of* (1) *starting from $X_0 \sim \nu(0)$.*
3. *It holds that $\nu(T) = \nu(0)$.*

In this case, we can obviously extend $(\nu(t))$ for $t \in \mathbb{R}$ by periodicity. Considering now the solution $(X_t)_{t\geq 0}$ of (1) defined for $t \geq 0$ it remains true that $\nu(t) = \mathcal{L}(X_t)$ for any $t \geq 0$. Moreover, we can also consider the solution of (1) defined on $[t_0, +\infty)$ for any $t_0 \in \mathbb{R}$ with initial condition $\mathcal{L}(X_{t_0}) = \nu(t_0)$.

We study the existence of periodic solutions $t \mapsto \mathcal{L}(X_t)$ where (X_t) is the solution of (1), near a non-stable invariant measure $\nu^\infty_{\alpha_0}$. Obviously, one necessary condition is that the criterion (17) is not satisfied in α_0. We assume that (17) is not satisfied in the following way

Assumption 8. *Assume that there exists $\alpha_0 > 0$ and $\tau_0 > 0$ such that*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) = 1 \quad \text{and} \quad \frac{d}{dz}\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) \neq 0.$$

Assumption 9 (Non-resonance condition)**.** *Assume that for all $n \in \mathbb{Z}\backslash\{-1, 1\}$,*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{in}{\tau_0}) \neq 1.$$

Remark 10 (Local uniqueness of the invariant measure in the neighborhood of α_0)**.** *Under Assumption 9, we have in particular $J(\alpha_0)\widehat{\Theta}_\alpha(0) \neq 1$ and so, by Lemma 6*

$$J'(\alpha_0) \neq 0.$$

Recall that the values of α such that ν^∞_α is an invariant measure of (1) *are precisely the solutions of $J(\alpha) = J$. So, in the neighborhood of $\alpha = \alpha_0$, the invariant measure of* (1) *is (locally) unique.*

Lemma 11. *Under Assumption 8, there exists $\eta_0, \varrho_0 > 0$ and a function $\mathfrak{Z}_0 \in \mathcal{C}^1((\alpha_0 - \eta_0, \alpha_0 + \eta_0); \mathbb{C})$ with $\mathfrak{Z}_0(\alpha_0) = \frac{i}{\tau_0}$ such that for all $z \in \mathbb{C}$ with $|z - \frac{i}{\tau_0}| < \varrho_0$ and for all $\alpha > 0$ with $|\alpha - \alpha_0| < \eta_0$ we have*

$$J(\alpha)\widehat{\Theta}_\alpha(z) = 1 \iff z = \mathfrak{Z}_0(\alpha). \tag{22}$$

Proof. It suffices to apply the Implicit Function Theorem to $(\alpha, z) \mapsto J(\alpha)\widehat{\Theta}_\alpha(z) - 1$. □

Assumption 12. *Assume that $\alpha \mapsto \mathfrak{Z}_0(\alpha)$ crosses the imaginary part with non-vanishing speed, that is*

$$\Re\mathfrak{Z}_0'(\alpha_0) \neq 0, \quad \text{where} \quad \mathfrak{Z}_0'(\alpha) = \frac{d}{d\alpha}\mathfrak{Z}_0(\alpha).$$

Remark 13. *Using* (22)*, Assumption 12 is equivalent to*

$$\Re\left(\frac{\frac{\partial}{\partial\alpha}\left(J(\alpha)\widehat{\Theta}_\alpha\right)\Big|_{\alpha=\alpha_0}(\frac{i}{\tau_0})}{J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0})}\right) \neq 0.$$

Our main result is the following.

Theorem 14. *Consider b, f and $\alpha_0, \tau_0 > 0$ such that Assumptions 2, 3, 4, 8, 9 and 12 hold. Then, there exists a family of $2\pi\tau_v$-periodic solutions of* (1)*, parametrized by $v \in (-v_0, v_0)$, for some $v_0 > 0$. More precisely, there exists a continuous curve $\{(\nu_v(\cdot), \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ such that*

1. *For all $v \in (-v_0, v_0)$, $(\nu_v(t))_{t\in\mathbb{R}}$ is a $2\pi\tau_v$-periodic solution of* (1) *with $J = J(\alpha_v)$.*
2. *The curve passes through $(\nu^\infty_{\alpha_0}, \alpha_0, \tau_0)$ at $v = 0$. In particular we have for all $t \in \mathbb{R}$, $\nu_0(t) \equiv \nu^\infty_{\alpha_0}$.*
3. *The "periodic current" $\boldsymbol{a}_v$, defined by*
$$t \mapsto a_v(t) := J(\alpha_v) \int_{\mathbb{R}_+} f(x)\nu_v(t, dx), \tag{23}$$
is continuous and $2\pi\tau_v$-periodic. Moreover, its mean over one period is α_v:
$$\frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)du = \alpha_v.$$
4. *Furthermore, v is the amplitude of the first harmonic of a_v, that is for all $v \in (-v_0, v_0)$*
$$\frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)\cos(u/\tau_v)du = v \quad \text{and} \quad \frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)\sin(u/\tau_v)du = 0.$$

Every other periodic solutions in a neighborhood of $\nu^\infty_{\alpha_0}$ is obtained from a phase-shift of one such ν_v. More precisely, there exists small enough constants $\epsilon_0, \epsilon_1 > 0$ (only depending on b, f, α_0 and τ_0) such that if $(\nu(t))_{t\in\mathbb{R}}$ is any $2\pi\tau$-periodic solution of (1) *for some value of $J > 0$ such that*
$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t\in[0,2\pi\tau]} \left| J \int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0 \right| < \epsilon_1,$$
then there exists a shift $\theta \in [0, 2\pi\tau)$ and $v \in (-v_0, v_0)$ such that $J = J(\alpha_v)$ and
$$\forall t \in \mathbb{R}, \quad \nu(t) \equiv \nu_v(t + \theta).$$

Remark 15. *Given the "periodic current" $\boldsymbol{a}_\nu$ defined by* (23)*, the shape of the solution is known explicitly: for all $v \in (-v_0, v_0)$, it holds that*
$$\nu_v = \tilde{\nu}_{\boldsymbol{a}_v},$$
where $\tilde{\nu}_{\boldsymbol{a}_v}$, defined by (46) *below, is known explicitly in terms of b, f and $\boldsymbol{a}_v$.*

Notation 16. *For $T > 0$, we denote by C^0_T the space of continuous and T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$ and by $C^{0,0}_T$ the subspace of centred functions*
$$C^{0,0}_T := \{h \in C^0_T, \quad \int_0^T h(t)dt = 0\}.$$

We now give an outline of the proof of Theorem 14. The proof is divided in two main parts.

The first part is devoted to the study of an isolated neuron subject to a periodic external current. That is, given $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi\tau}$, we study the jump rate of an isolated neuron driven by $\boldsymbol{a}$. We give in Section 3.1 estimates on the kernels $K_{\boldsymbol{a}}$ and $H_{\boldsymbol{a}}$. We want to characterize the "asymptotic" jump rate of a neuron driven by this external periodic current, that is
$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \lim_{k\in\mathbb{N},\ k\to\infty} r_{\boldsymbol{a}}(t, -2\pi k\tau).$$

In order to characterize such limit $\rho_{\boldsymbol{a}}$, we introduce in Section 3.2 a discrete-time Markov Chain corresponding to the phases of the successive spikes of the neuron driven by $\boldsymbol{a}$. We prove that this Markov Chain has a unique invariant measure, which is proportional to $\rho_{\boldsymbol{a}}$. This serves as a definition of $\rho_{\boldsymbol{a}}$. Given this periodic jump rate $\rho_{\boldsymbol{a}} \in C^0_{2\pi\tau}$, we give in Section 3.3 an explicit description of the associated time-periodic probability densities, that we denote $(\tilde{\nu}_{\boldsymbol{a}}(t))_{t\in[0,2\pi\tau]}$. Consequently, to find a $2\pi\tau$-periodic solution of (1), it is equivalent to find $\boldsymbol{a} \in C^0_{2\pi\tau}$ such that

$$\boldsymbol{a} = J\rho_{\boldsymbol{a}}. \tag{24}$$

One classical difficulty with Hopf bifurcation is that the period $2\pi\tau$ itself is unknown: τ varies when the interaction parameter J varies. To address this problem, we make in Section 3.4 a change of time to only consider 2π-periodic functions. We define

$$\forall \boldsymbol{d} \in C^0_{2\pi}, \forall \tau > 0, \quad \rho_{\boldsymbol{d},\tau} = \mathcal{T}_\tau(\rho_{\mathcal{T}_{1/\tau}(\boldsymbol{d})}), \quad \text{with} \quad \forall t \geq 0,\ \mathcal{T}_\tau(\boldsymbol{d})(t) := d(\tau t). \tag{25}$$

We shall see that this change of time has a simple probabilistic interpretation by scaling b, f and $\boldsymbol{d}$ appropriately. In Section 3.5, we prove that the function $C^0_{2\pi} \times \mathbb{R}^*_+ \ni (\boldsymbol{d}, \tau) \mapsto \rho_{\boldsymbol{d},\tau} \in C^0_{2\pi}$ is $\mathcal{C}^2$-Fréchet differentiable. Furthermore, if the mean over one period of $\boldsymbol{d}$ is α, that is if $\boldsymbol{d} = \alpha + h$ for some $h \in C^{0,0}_{2\pi}$, we prove that the mean number of spikes over one period only depends on α, namely

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha+h,\tau}(u)du = \gamma(\alpha). \tag{26}$$

In the second part of the proof, we find self-consistent periodic solutions using the Lyapunov-Schmidt method. We introduce in Section 3.6 the following functional

$$C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \ni (h, \alpha, \tau) \mapsto G(h, \alpha, \tau) := (\alpha + h) - J(\alpha)\rho_{\alpha+h,\tau}.$$

Using (26), this functional takes values in $C^{0,0}_{2\pi}$. The roots of G, described by Proposition 32, match with the periodic solutions of (1). For instance if $G(h, \alpha, \tau) = 0$, we set $\boldsymbol{a} := \mathcal{T}_\tau(\alpha + h)$ which solves (24) with $J = J(\alpha)$ and so it can be used to define a periodic solution of (1). Conversely, to any periodic solution of (1), we can associate a root of G. So Theorem 14 is equivalent to Proposition 32. Sections 3.7, 3.8, 3.9 and 3.10 are then devoted to the proof of Proposition 32. In Section 3.7, we prove that the linear operator $D_hG(0, \alpha, \tau)$ can be written using a convolution involving Θ_α, given by (14). We then follow the method of [Kie12, Ch. I.8]. In Section 3.8, we study the range and the kernel of $D_hG(0, \alpha_0, \tau_0)$: we prove that under the spectral Assumptions 8 and 9, $D_hG(0, \alpha_0, \tau_0)$ is a Fredholm operator of index zero, with a kernel of dimension two. The problem of finding the roots of G is a priori of infinite dimension (h belongs to $C^{0,0}_{2\pi}$). In Section 3.9 we apply the Lyapunov-Schmidt method to obtain an equivalent problem of dimension two. Finally in Section 3.10 we study the reduced 2D-problem.

3 Proof of Theorem 14

3.1 Preliminaries

Let $T > 0$, $s \in \mathbb{R}$ and $\boldsymbol{a} \in C^0_T$ such that

$$\inf_{t\in[0,T]} a_t > -b(0). \tag{27}$$

For $x \geq 0$, we consider $\varphi^{\boldsymbol{a}}_{t,s}(x)$ the solution of the ODE

$$\begin{aligned}\frac{d}{dt}\varphi^{\boldsymbol{a}}_{t,s}(x) &= b(\varphi^{\boldsymbol{a}}_{t,s}(x)) + a_t \\ \varphi^{\boldsymbol{a}}_{s,s}(x) &= x.\end{aligned} \tag{28}$$

By Assumption 2, this ODE has a unique solution. Moreover, the kernels $H^\nu_{\boldsymbol{a}}(t,s)$ and $K^\nu_{\boldsymbol{a}}(t,s)$, defined by (5), have an explicit expression in term of the flow

$$H^\nu_{\boldsymbol{a}}(t,s) = \int_{\mathbb{R}_+} \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx), \tag{29}$$

$$K^\nu_{\boldsymbol{a}}(t,s) = \int_{\mathbb{R}_+} f(\varphi^{\boldsymbol{a}}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx). \tag{30}$$

The function $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ belongs to $\mathcal{C}^1((-\infty,t];\mathbb{R}_+)$ and

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) = -\left[b(0)+a_s\right]\exp\left(\int_s^t b'(\varphi^{\boldsymbol{a}}_{\theta,s}(0))d\theta\right). \tag{31}$$

In particular, under the assumption (27), $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing on $(-\infty,t]$, for all t. Define then

$$\sigma_{\boldsymbol{a}}(t) := \lim_{s\to-\infty} \varphi^{\boldsymbol{a}}_{t,s}(0) \in \mathbb{R}^*_+ \cup \{+\infty\}. \tag{32}$$

Given $d \in C^0_T$ and $\eta > 0$, we define the following open balls of C^0_T:

$$B^T_\eta(d) := \{a \in C^0_T, \quad \sup_{t\in[0,T]} |a_t - d_t| < \eta\}. \tag{33}$$

Lemma 17. *Let $T > 0$ and $b : \mathbb{R}_+ \to \mathbb{R}$ such that Assumption 2 holds. Let $\alpha_0 > 0$ satisfying Assumption 4. There exists $\eta_0 > 0$ such that for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

1. *If $\sigma_{\alpha_0} = \infty$, then for all $t \in [0,T]$, $\sigma_{\boldsymbol{a}}(t) = +\infty$.*
2. *If $\sigma_{\alpha_0} < \infty$, the function $t \mapsto \sigma_{\boldsymbol{a}}(t)$ belongs to C^0_T and*

$$\inf_{\boldsymbol{a}\in B^T_{\eta_0}(\alpha_0)} \inf_{t\in[0,T]} \sigma_{\boldsymbol{a}}(t) > 0.$$

Proof. Assume first that $\sigma_{\alpha_0} = \infty$, and let $\eta_0 := \frac{1}{2}\inf_{x\geq 0} b(x) + \alpha_0$, which is strictly positive by assumption. Then it holds that

$$\inf_{t\geq 0}\inf_{x\geq 0} b(x) + a_t \geq \frac{\eta_0}{2},$$

and so

$$\varphi^{\boldsymbol{a}}_{t,s}(0) \geq \frac{\eta_0}{2}(t-s). \tag{34}$$

Letting s going to $-\infty$, we deduce that $\sigma_{\boldsymbol{a}}(t) = +\infty$.
Assume now that $\sigma_{\alpha_0} < \infty$. Using (15), we can apply the Implicit Function Theorem to the function

$$(x,\epsilon) \mapsto b(x) + \alpha_0 + \epsilon$$

at the point $(\sigma_{\alpha_0}, 0)$. This gives the existence of $\epsilon_0 > 0$ and the existence of a function $\epsilon \mapsto \sigma_{\alpha_0+\epsilon}$, which belongs to $\mathcal{C}^1([0,\epsilon_0];\mathbb{R}^*_+)$, such that

$$\forall\epsilon\in[0,\epsilon_0], \quad \sigma_{\alpha_0} \leq \sigma_{\alpha_0+\epsilon} < \infty \quad \text{ and } \quad \sigma_{\alpha_0+\epsilon} = \inf_{x\geq 0}\{b(x)+\alpha_0+\epsilon = 0\}.$$

We choose $\eta_0 := \frac{\epsilon_0}{2}$. Let $\boldsymbol{a} \in C^0_T$ such that $\sup_{t\in[0,T]}|a_t - \alpha_0| \leq \eta_0$. By the Grönwall inequality, we have

$$\forall t \geq s, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \leq \varphi^{\alpha_0+\epsilon_0}_{t,s}(0) \leq \sigma_{\alpha_0+\epsilon_0}.$$

In particular $\sigma_{\boldsymbol{a}}(t) < \infty$. We prove that this function is right-continuous in t. We fix $t \geq s$ and $\epsilon \in [0,\epsilon_0]$, we have

$$\begin{aligned}\varphi^{\boldsymbol{a}}_{t+\epsilon,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0) &= \varphi^{\boldsymbol{a}}_{t+\epsilon,t}(\varphi^{\boldsymbol{a}}_{t,s}(0)) - \varphi^{\boldsymbol{a}}_{t,s}(0)\\ &= \int_t^{t+\epsilon} b(\varphi^{\boldsymbol{a}}_{t+u,s}(0))du + \int_t^{t+\epsilon} a_u du.\end{aligned}$$

So if $A_0 := \sup_{x\in[0,\sigma_{\alpha_0+\epsilon_0}]} |b(x)| < \infty$, we deduce that

$$|\varphi^{\boldsymbol{a}}_{t+\epsilon,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0)| \leq (A_0 + ||\boldsymbol{a}||_\infty)\epsilon. \tag{35}$$

Letting s to $-\infty$ we deduce that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is right-continuous. Left-continuity is proved similarly. Using $\varphi^{\boldsymbol{a}}_{t+T,s+T}(0) = \varphi^{\boldsymbol{a}}_{t,s}(0)$, we deduce that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic. Finally, because $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing, and takes value 0 when $s = t$, we deduce that $\sigma_{\boldsymbol{a}}(t) > 0$. More precisely, let

$$m_0 := -\min_{x\in[0,\sigma_{\alpha_0+\epsilon_0}]} b'(x).$$

It holds that $m_0 > 0$. Moreover, using (31), we deduce that

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) \leq -(b(0) + \alpha_0 - \eta_0)e^{-m_0(t-s)},$$

and so

$$\forall s \leq t, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \geq (b(0) + \alpha_0 - \eta_0)\frac{1 - e^{-m_0(t-s)}}{m_0}. \tag{36}$$

It ends the proof. □

Inspecting the proof of Lemma 17, we deduce that

Lemma 18. *Grant Assumptions 2 and 3. Let $\alpha_0 > 0$ such that Assumption 4 holds. There exists $\lambda_0, \eta_0, s_0 > 0$ (only depending on α_0 and b) such that for all $T > 0$, for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

$$\forall t, s, \quad t - s \geq s_0 \implies \varphi^{\boldsymbol{a}}_{t,s}(0) \geq \lambda_0.$$

Moreover, if $\sigma_{\alpha_0} = \infty$, λ_0 can be chosen arbitrarily large. Finally, it holds that

$$\sup_{T>0} \sup_{\boldsymbol{a}\in B^T_{\eta_0}(\alpha_0)} \sup_{t\geq s} \left[H_{\boldsymbol{a}}(t,s) + K_{\boldsymbol{a}}(t,s)\right] e^{f(\lambda_0)(t-s)} < \infty.$$

Proof. Case $\sigma_{\alpha_0} < \infty$: the lower bound of the flow follows from (36). The bounds on H and K then follow directly from their explicit expressions (29) and (30) and the upper bound $f(\varphi^{\boldsymbol{a}}_{t,s}(0)) \leq f(\sigma_{\alpha_0+\epsilon_0})$.
Case $\sigma_{\alpha_0} = \infty$: the lower bound of the flow is a consequence of (34). Similarly, the bound on H follows from (29). Note moreover that Assumption 3.4 and the global Lipschitz property of b (say with constant L) ensure the existence of a constant C such that

$$f(\varphi^{\boldsymbol{a}}_{t,s}(0)) \leq Ce^{Lp(t-s)}.$$

The bound on K follows. □

3.2 Study of the non-homogeneous linear equation

In this section, we study the asymptotic jump rate of an "isolated" neuron driven by a periodic continuous function. Grant Assumptions 2, 3 and let $\alpha_0 > 0$ such that Assumption 4 holds. Let $\lambda_0, \eta_0 > 0$ be given by Lemma 18 and $T > 0$. Consider $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$. Following the terminology of [CTV20], we say that $\boldsymbol{a}$ is the "external current". Let $r_{\boldsymbol{a}}$ be the solution of the Volterra equation $r_{\boldsymbol{a}} = K_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r_{\boldsymbol{a}}$. We consider the following limit

$$\forall t \in [0,T], \quad \rho_{\boldsymbol{a}}(t) = \lim_{k\to-\infty} r_{\boldsymbol{a}}(t, -kT).$$

The goal of this section is to show that $\rho_{\boldsymbol{a}}$ is well defined and to study some of its properties. First, (6) and (7) write

$$\forall t \in \mathbb{R}, \quad r_{\boldsymbol{a}}(t,-kT) = K_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^t K_{\boldsymbol{a}}(t,s)r_{\boldsymbol{a}}(s,-kT)ds,$$
$$1 = H_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^t H_{\boldsymbol{a}}(t,s)r_{\boldsymbol{a}}(s,-kT)ds.$$

Letting $k \to \infty$, we find that $\rho_{\boldsymbol{a}}$ has to solve

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{37}$$

$$1 = \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{38}$$

Note that if $\rho_{\boldsymbol{a}}$ is a solution of (37), then it automatically holds that the function $t \mapsto \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds$ is constant (its derivative is null). In Lemma 20 below, we prove that the solutions of equation (37) form a linear space of dimension 1. Consequently (37) together with (38) have a unique solution: this will serve as the definition of $\rho_{\boldsymbol{a}}$.

A probabilistic interpretation of (37) **and** (38)

Let x be a T-periodic solution of (37). We have for all $t \in [0,T]$

$$\begin{aligned} x(t) &= \int_{-\infty}^{T} K_{\boldsymbol{a}}(t,s)x(s)ds, \quad (\text{recall } K_{\boldsymbol{a}}(t,s) = 0 \text{ for } s > t) \\ &= \sum_{k\geq 0} \int_{-kT}^{T-kT} K_{\boldsymbol{a}}(t,s)x(s)ds \\ &= \sum_{k\geq 0} \int_{0}^{T} K_{\boldsymbol{a}}(t,u-kT)x(u)du \quad (\text{by the change of variable } u = s+kT). \end{aligned}$$

Define for all $t,s \in [0,T]$

$$K^T_{\boldsymbol{a}}(t,s) := \sum_{k\geq 0} K_{\boldsymbol{a}}(t,s-kT).$$

Note that by Lemma 18 we have normal convergence:

$$\forall t,s \in [0,T], \quad K_{\boldsymbol{a}}(t,s-kT) \leq Ce^{-f(\lambda_0)k},$$

for some constant C only depending on b, f, $\alpha_0, \eta_0, \lambda_0$ and T. We deduce that x solves

$$x(t) = \int_0^T K^T_{\boldsymbol{a}}(t,s)x(s)ds. \tag{39}$$

Using that $\boldsymbol{a}$ is T-periodic, we have

$$\forall t \geq s, \quad K_{\boldsymbol{a}}(t+T,s+T) = K_{\boldsymbol{a}}(t,s). \tag{40}$$

Moreover, $K_{\boldsymbol{a}}$ is a probability density so

$$\forall s \in \mathbb{R}, \quad \int_s^{\infty} K_{\boldsymbol{a}}(t,s)dt = 1. \tag{41}$$

From (40) and (41), we deduce that

$$\forall s \in [0,T], \quad \int_0^T K^T_{\boldsymbol{a}}(t,s)dt = 1. \tag{42}$$

In view of (42), $K^T_{\boldsymbol{a}}(\cdot,s)$ can be seen as the transition probability kernel of a Markov Chain acting on the continuous space $[0,T]$. The interpretation of this Markov Chain is the following. Let $(Y^{\nu,\boldsymbol{a}}_t)_{t\geq 0}$ be the solution of (3), starting at time 0 with law ν and driven by the T-periodic current $\boldsymbol{a}$. Define $(\tau_i)_{i\geq 1}$ the times of the successive jumps of $(Y^{\boldsymbol{a},\nu}_t)_{t\geq 0}$. Let $\phi_i \in [0,T)$ and $\Delta_i \in \mathbb{N}$ be defined by:

$$\phi_i := \tau_i - \left\lfloor \frac{\tau_i}{T} \right\rfloor T, \quad \tau_{i+1} - \tau_i =: \Delta_{i+1}T + \phi_{i+1} - \phi_i. \tag{43}$$

That is, ϕ_i is the *phase* of the i-ith jump, while Δ_i is the number of "revolutions" between τ_{i-1} and τ_i:

$$\Delta_i = \#\{k \in \mathbb{N}, \quad kT \in [\tau_{i-1}, \tau_i)\}.$$

In other words, if one considers that a period is a "lap", Δ_i is the number of times we cross the start line of the lap between two spikes.

Then, $(\phi_i, \Delta_i)_{i\geq 0}$ is Markov, with a transition probability given by

$$\forall A \in \mathcal{B}([0,T]),\ \forall n \in \mathbb{N}, \quad \mathbb{P}(\phi_{i+1} \in A, \Delta_{i+1} = n | \phi_i) = \int_A K_{\boldsymbol{a}}(t + nT, \phi_i) dt.$$

In particular, $(\phi_i)_{i\geq 0}$ **is Markov, with a transition probability given by** $K^T_{\boldsymbol{a}}$. With some slight abuse of notations, we also write $K^T_{\boldsymbol{a}}$ for the linear operator which maps $y \in L^1([0,T])$ to

$$K^T_{\boldsymbol{a}}(y) := t \mapsto \int_0^T K^T_{\boldsymbol{a}}(t,s) y(s) ds \in L^1([0,T]). \tag{44}$$

Lemma 19. *Let* $\boldsymbol{a} \in C^0_T$. *The linear operator* $K^T_{\boldsymbol{a}} : L^1([0,T]) \to L^1([0,T])$ *is a compact operator. Moreover, if* $y \in L^1([0,T])$, *then* $K^T_{\boldsymbol{a}}(y) \in C^0_T$.

Proof. First, the function $[0,T]^2 \ni (t,s) \mapsto K^T_{\boldsymbol{a}}(t,s)$ is (uniformly) continuous. Let $\epsilon > 0$, there exists $\eta > 0$ such that

$$|t - t'| + |s - s'| \leq \eta \implies |K^T_{\boldsymbol{a}}(t,s) - K^T_{\boldsymbol{a}}(t',s')| \leq \epsilon.$$

It follows that

$$\left|K^T_{\boldsymbol{a}}(y)(t) - K^T_{\boldsymbol{a}}(y)(t')\right| \leq \int_0^T \left|K^T_{\boldsymbol{a}}(t,s) - K^T_{\boldsymbol{a}}(t',s)\right| |y(s)|\, ds \leq \epsilon ||y||_1,$$

and so the function $K^T_{\boldsymbol{a}}(y)$ is continuous. Note that

$$\forall s \in [0,T], \quad K^T_{\boldsymbol{a}}(T,s) = K^T_{\boldsymbol{a}}(0,s),$$

and so $K^T_{\boldsymbol{a}}(y)$ is T-periodic. This shows that $K^T_{\boldsymbol{a}}(y) \in C^0_T$. To prove that $K^T_{\boldsymbol{a}}$ is a compact operator, we use the Weierstrass approximation Theorem: there exists a sequence of polynomial functions $(t,s) \mapsto P_n(t,s)$ such that $\sup_{t,s\in[0,T]} \left|P_n(t,s) - K^T_{\boldsymbol{a}}(t,s)\right| \to_n 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, the linear operator $L^1([0,T]) \ni y \mapsto P_n(y) := t \mapsto \int_0^T P_n(t,s)y(s)ds$ is of finite-rank. Moreover, the sequence P_n converges to $K^T_{\boldsymbol{a}}$ for the norm operator, and so $K^T_{\boldsymbol{a}}$ is a compact operator (as the limit of finite-rank operators, see [Bre11, Ch. 6]). $\square$

Lemma 20. *Let* $\boldsymbol{a} \in C^0_T$. *The Markov Chain* $(\phi_i)_{i\geq 0}$ *with transition probability kernel* $K^T_{\boldsymbol{a}}$ *has a unique invariant probability measure* $\pi_{\boldsymbol{a}} \in C^0_T$. *Consequently, the solutions of* (37) *in* C^0_T *span a vector space of dimension* 1.

Proof. The proof follows directly from the Krein–Rutman Theorem, which is a generalization of the Perron-Frobenius Theorem for compact operators. We give the proof for the sake of completeness, and because some of its elements will be reused later. Let $(K^T_{\boldsymbol{a}})' : L^\infty([0,T]) \to L^\infty([0,T])$ be the dual operator of $K^T_{\boldsymbol{a}}$. We have:

$$\forall v \in L^\infty([0,T]), \quad (K^T_{\boldsymbol{a}})'(v) = t \mapsto \int_0^T K^T_{\boldsymbol{a}}(s,t) v(s) ds.$$

From (42), we deduce that 1 is an eigenvalue of $(K^T_{\boldsymbol{a}})'$ (its associated eigenvector is $\mathbf{1}$, the constant function equal to 1). By the Fredholm alternative, we have $\dim N(I - K^T_{\boldsymbol{a}}) = \dim N(I - (K^T_{\boldsymbol{a}})')$ and so there exists $\pi \in L^1([0,T])$ such that:

$$\pi = K^T_{\boldsymbol{a}}(\pi),\ ||\pi||_1 = 1.$$

We now prove that π can be chosen positive. Let $\delta := \inf_{t,s\in[0,T]} K^T_{\boldsymbol{a}}(t,s)$. $K^T_{\boldsymbol{a}}$ is positive and continuous on $[0,T]^2$ so $\delta > 0$. We write π_+ for the positive part of π and π_- for its negative part and define $\beta := \min(||\pi_+||_1, ||\pi_-||_1)$. We have $K^T_{\boldsymbol{a}}(\pi_+)(t) \geq \delta\beta T$ and $K^T_{\boldsymbol{a}}(\pi_-)(t) \geq \delta\beta T$. Consequently

$$\begin{aligned} ||K^T_{\boldsymbol{a}}(\pi)||_1 =& ||K^T_{\boldsymbol{a}}(\pi_+) - K^T_{\boldsymbol{a}}(\pi_-)||_1 \\ \leq& ||K^T_{\boldsymbol{a}}(\pi_+) - \delta\beta T||_1 + ||K^T_{\boldsymbol{a}}(\pi_-) - \delta\beta T||_1 \\ \leq& ||\pi||_1 - 2\delta\beta T. \end{aligned}$$

But the identity $K^T_{\boldsymbol{a}}(\pi) = \pi$ implies that $\beta = 0$ and so either π_+ or π_- is null. So π has a constant sign and may be chosen positive. Note moreover that

$$\pi(t) = \int_0^T K^T_{\boldsymbol{a}}(t,s)\pi(s)ds \geq \delta \int_0^T \pi(s)ds \geq \delta.$$

Finally, if π_1 and π_2 are two non-negative solutions of (39) with $||\pi_1||_1 = ||\pi_2||_1 = 1$, then $\pi_3 := \pi_1 - \pi_2$ also solves (39) and has a constant sign. Consequently, $||\pi_3||_1 = |||\pi_1||_1 - ||\pi_2||_1| = 0$ and we deduce that $\pi_3 = 0$, proving that the space of solutions in $L^1([0,T])$ of (39) is of dimension 1. Finally Lemma 19 gives the continuity of π and $\pi(T) = \pi(0)$. Consequently π can be extended to C^0_T and solves (37). This ends the proof. $\square$

We define for all $\theta \in \mathbb{R}$ the following shift operator

$$\begin{array}{rccc} S_\theta: & C^0_T & \to & C^0_T \\ & x & \mapsto & (x(t+\theta))_t. \end{array}$$

Corollary 21. *Given $\boldsymbol{a} \in C^0_T$, equations* (37) *and* (38) *have a unique solution $\rho_{\boldsymbol{a}} \in C^0_T$. Moreover, it holds that for all $\theta \in \mathbb{R}$,*

$$\rho_{S_\theta \boldsymbol{a}} = S_\theta \rho_{\boldsymbol{a}}. \tag{45}$$

Proof. By Lemma 20, the solution $\rho_{\boldsymbol{a}}$ of equations (37) and (38) is

$$\rho_{\boldsymbol{a}} = \frac{\pi_{\boldsymbol{a}}}{c_{\boldsymbol{a}}},$$

where $\pi_{\boldsymbol{a}}$ is the invariant measure (on $[0,T]$) of the Markov Chain with transition probability kernel $K^T_{\boldsymbol{a}}$ and $c_{\boldsymbol{a}}$ is given by

$$c_{\boldsymbol{a}} := \int_{-\infty}^t H_{\boldsymbol{a}}(t,s)\pi_{\boldsymbol{a}}(s)ds.$$

Note that $c_{\boldsymbol{a}}$ is constant in time. Define for all $t,s \in [0,T]$:

$$H^T_{\boldsymbol{a}}(t,s) := \sum_{k\geq 0} H_{\boldsymbol{a}}(t, s-kT).$$

We have $c_{\boldsymbol{a}} = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}})$. Moreover, we have

$$\forall t,s,\theta \in \mathbb{R}, \quad \varphi^{S_\theta \boldsymbol{a}}_{t,s}(0) = \varphi^{\boldsymbol{a}}_{t+\theta,s+\theta}(0),$$

because both sides satisfy the same ODE with the same initial condition at $t = s$. We deduce from (29) and (30) that

$$H_{S_\theta \boldsymbol{a}}(t,s) = H_{\boldsymbol{a}}(t+\theta, s+\theta), \quad K_{S_\theta \boldsymbol{a}}(t,s) = K_{\boldsymbol{a}}(t+\theta, s+\theta).$$

So $S_\theta \rho_{\boldsymbol{a}}$ solves (37) and (38), where the kernels are replaced by $K_{S_\theta \boldsymbol{a}}$ and $H_{S_\theta \boldsymbol{a}}$. By uniqueness it follows that $\rho_{S_\theta \boldsymbol{a}} = S_\theta \rho_{\boldsymbol{a}}$. $\square$

Remark 22. *Using that* $\int_0^T \pi_{\boldsymbol{a}}(s)ds = 1$*, we find that the average number of spikes over one period* $[0,T]$ *is given by*

$$\frac{1}{T}\int_0^T \rho_{\boldsymbol{a}}(s)ds = \frac{1}{c_{\boldsymbol{a}}T}.$$

The probabilistic interpretation of $c_{\boldsymbol{a}}$ *is the following: remembering the Markov chain defined by* (43)*, we have*

$$\mathbb{P}(\Delta_{i+1} > k|\phi_i) = H_{\boldsymbol{a}}((k+1)T, \phi_i),$$

and so, if $\mathcal{L}(\phi_i) = \pi_{\boldsymbol{a}}$*, we deduce that*

$$\mathbb{E}\,\Delta_{i+1} = \mathbb{E}\,\mathbb{E}(\Delta_{i+1}|\phi_i) = \mathbb{E}\left[\sum_{k\geq 0}\mathbb{P}(\Delta_{i+1} > k|\phi_i)\right] = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}}) = c_{\boldsymbol{a}}.$$

In other words, $c_{\boldsymbol{a}}$ *is the expected number of "revolutions" between two successive spikes, assuming the phase of each spikes follow its invariant measure* $\pi_{\boldsymbol{a}}$*.We shall see in Proposition 31 that* $c_{\boldsymbol{a}}$ *only depends on the mean of* $\boldsymbol{a}$*. Furthermore, it holds that for* $\boldsymbol{a} \equiv \alpha > 0$

$$c_\alpha = H^T_\alpha(1/T) = \frac{1}{T}\int_0^\infty H_\alpha(t)dt = \frac{1}{T\gamma(\alpha)},$$

and so for all t*,* $\rho_\alpha(t) = \gamma(\alpha)$*.*

3.3 Shape of the solutions

Let $\boldsymbol{a} \in C^0_T$ such that (27) holds. Let $\sigma_{\boldsymbol{a}}(t)$ be defined by (32), such that $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is a bijection from $(-\infty, t]$ to $[0, \sigma_{\boldsymbol{a}}(t))$. We denote by $x \mapsto \beta^{\boldsymbol{a}}_t(x)$ its inverse. Note that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic and

$$\forall t \in \mathbb{R}, \forall x \in [0, \sigma_{\boldsymbol{a}}(t)), \quad \beta^{\boldsymbol{a}}_{t+T}(x) = \beta^{\boldsymbol{a}}_t(x) + T.$$

Using that $\varphi^{\boldsymbol{a}}_{t,t}(0) = 0$, we have $\beta^{\boldsymbol{a}}_t(0) = t$.

Notation 23. *Given* $\boldsymbol{a} \in C^0_T$*, we define for all* $t \in \mathbb{R}$

$$\tilde{\nu}_{\boldsymbol{a}}(t,x) := \frac{\rho_{\boldsymbol{a}}(\beta^{\boldsymbol{a}}_t(x))}{b(0) + a(\beta^{\boldsymbol{a}}_t(x))}\exp\left(-\int_{\beta^{\boldsymbol{a}}_t(x)}^t (f+b')(\varphi^{\boldsymbol{a}}_{\theta,\beta^{\boldsymbol{a}}_t(x)}(0))d\theta\right)\mathbb{1}_{[0,\sigma_{\boldsymbol{a}}(t))}(x), \tag{46}$$

where $\rho_{\boldsymbol{a}}$ *is the unique solution of the equations* (37) *and* (38)*.*

By the change of variables $u = \beta^{\boldsymbol{a}}_t(x)$, one obtains that for any non-negative measurable test function g

$$\int_0^\infty g(x)\tilde{\nu}_{\boldsymbol{a}}(t,x)dx = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du. \tag{47}$$

Note moreover than when $\boldsymbol{a}$ is constant and equal to $\alpha > 0$ ($\boldsymbol{a} \equiv \alpha$), (46) matches with the definition of the invariant measure ν^∞_α given by (9):

$$\forall t \in \mathbb{R}, \quad \sigma_\alpha(t) = \sigma_\alpha \quad \text{and} \quad \tilde{\nu}_\alpha(t) = \nu^\infty_\alpha.$$

The main result of this section is

Proposition 24. *Let* $\boldsymbol{a} \in C^0_T$ *such that* $\inf_{t\in\mathbb{R}} a_t > -b(0)$*. It holds that* $(\tilde{\nu}_{\boldsymbol{a}}(t,\cdot))_t$ *is the unique* T*-periodic solution of* (3)*.*

Proof. **Existence**. We first prove that $\tilde{\nu}_{\boldsymbol{a}}(t,\cdot)$ is indeed a T-periodic solution. We follow the same strategy that [CTV20, Prop. 26]. First note that, by (47), one has

$$\int_0^\infty f(x)\tilde{\nu}_{\boldsymbol{a}}(t,x)dx = \int_{-\infty}^t K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du = \rho_{\boldsymbol{a}}(t).$$

Consider $(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)})$ be the solution of (3) starting with law $\tilde{\nu}_{\boldsymbol{a}}(0)$ at time $t=0$ and let $r_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t) = \mathbb{E}\, f(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)})$.
Claim: It holds that for all $t \geq 0$, $r_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t) = \rho_{\boldsymbol{a}}(t)$.

Proof of the Claim. Recall that $r_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t)$ is the unique solution of the Volterra equation

$$r_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)} = K_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)} + K_{\boldsymbol{a}} * r_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}.$$

So, to prove the claim is suffices to show that $\rho_{\boldsymbol{a}}$ also solves this equation. For all $u \leq 0 \leq t$, one has

$$K_{\boldsymbol{a}}^{\varphi_{0,u}^{\boldsymbol{a}}(0)}(t,0)H_{\boldsymbol{a}}(0,u) = K_{\boldsymbol{a}}(t,u).$$

Consequently, we deduce from (47) that

$$K_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t,0) = \int_{-\infty}^0 K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du.$$

So

$$\rho_{\boldsymbol{a}}(t) = \int_{-\infty}^t K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du = K_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t,0) + \int_0^t K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du,$$

and the conclusion follows. □

Finally, using [CTV20, Prop. 19] and the claim, we deduce that for any non-negative measurable function g

$$\mathbb{E}\, g(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}) = \int_0^t g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du + \int_0^\infty g(\varphi_{t,0}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,0)\tilde{\nu}_{\boldsymbol{a}}(0,x)dx.$$

By (47), the second term is equal to

$$\int_{-\infty}^0 g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du,$$

and so

$$\forall t \geq 0, \quad \mathbb{E}\, g(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}) = \int_{-\infty}^t g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du \overset{(47)}{=} \int_0^\infty g(x)\tilde{\nu}_{\boldsymbol{a}}(t,x)dx.$$

This ends the proof of the existence.
Uniqueness. Consider $(\nu(t))_{t\in[0,T]}$ a T-periodic solution of (3) and define $\rho(t) = \mathbb{E}\, f(Y_{t,0}^{\boldsymbol{a},\nu(0)})$. The function ρ is T-periodic. Moreover, it holds that for all $k \geq 0$, $\rho(t) = \mathbb{E}\, f(Y_{t,-kT}^{\boldsymbol{a},\nu(0)})$ and so (6) and (7) yields

$$\rho(t) = K_{\boldsymbol{a}}^{\nu(0)}(t,-kT) + \int_{-kT}^t K_{\boldsymbol{a}}(t,u)\rho(u)du$$
$$1 = H_{\boldsymbol{a}}^{\nu(0)}(t,-kT) + \int_{-kT}^t H_{\boldsymbol{a}}(t,u)\rho(u)du.$$

Letting k goes to infinity, we deduce that ρ solves (37) and (38). By uniqueness, we deduce that for all t, $\rho(t) = \rho_{\boldsymbol{a}}(t)$ (and so ρ is continuous). Finally define τ_t the time of the last spike of

$Y^{\boldsymbol{a},\nu(0)}_{t,-kT}$ before t (with the convention that $\tau_t = -kT$ if there is no spike between $-kT$ and t). The law of τ_t is

$$\mathcal{L}(\tau_t)(du) = \delta_{-kT}(du) H_{\boldsymbol{a}}(t,-kT) + \rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u)du.$$

Consequently, for any non-negative test function g:

$$\begin{aligned}\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}) &= \mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT} \mathbb{1}_{\tau_t=-kT}) + \mathbb{E}\, g(\varphi^{\boldsymbol{a}}_{t,\tau_t}(0))\mathbb{1}_{\tau_t\in(-kT,t]} \\ &= \int_0^\infty g(\varphi^{\boldsymbol{a}}_{t,-kT}(x)) H^x_{\boldsymbol{a}}(t,-kT)\nu(0)(dx) + \int_{-kT}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u)du.\end{aligned}$$

Using that $\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}) = \mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0})$ and letting again k to infinity we deduce that

$$\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0}) = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u)du.$$

So for all t, $\nu(t) \equiv \tilde{\nu}_{\boldsymbol{a}}(t)$. □

3.4 Reduction to 2π-periodic functions

Convention: For now on, we prefer to work with the *reduced period* τ, such that

$$T =: 2\pi\tau, \quad \tau > 0.$$

Consider $\boldsymbol{d} \in C^0_{2\pi\tau}$ and let $\boldsymbol{a}$ be the 2π-periodic function defined by:

$$\forall t \in \mathbb{R}, \quad a(t) := d(\tau t).$$

We define

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a},\tau}(t) := \rho_{\boldsymbol{d}}(\tau t),$$

where $\rho_{\boldsymbol{d}}$ is the unique solution of (37) and (38) (with kernels $K_{\boldsymbol{d}}$ and $H_{\boldsymbol{d}}$). Because $\rho_{\boldsymbol{d}}$ is $2\pi\tau$-periodic, $\rho_{\boldsymbol{a},\tau}$ is 2π-periodic. Note that when $\boldsymbol{a} \equiv \alpha$ is constant we have

$$\forall \tau > 0, \forall t \in \mathbb{R}, \quad \rho_{\alpha,\tau}(t) = \gamma(\alpha). \tag{48}$$

To better understand how $\rho_{\boldsymbol{a},\tau}$ depends on τ, consider $(Y^{\boldsymbol{d},\nu}_{t,s})$ the solution of (3), starting with law ν and driven by $\boldsymbol{d}$. Note that for all $t \geq s$

$$\begin{aligned}Y^{\boldsymbol{d},\nu}_{\tau t,\tau s} &= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_{\tau s}^{\tau t} b(Y^{\boldsymbol{d},\nu}_{u,\tau s})du + \int_{\tau s}^{\tau t} d_u du - \int_{\tau s}^{\tau t}\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{u-,\tau s}\mathbb{1}_{\{\tau z \leq \tau f(Y^{\boldsymbol{d},\nu}_{u-,\tau s})\}}\mathbf{N}(du,dz) \\ &= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_s^t \tau b(Y^{\boldsymbol{d},\nu}_{\tau u,\tau s})du + \int_s^t \tau a_u du - \int_s^t\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s}\mathbb{1}_{\{z \leq \tau f(Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s})\}}\tilde{\mathbf{N}}(du,dz).\end{aligned}$$

Here, $\tilde{\mathbf{N}} := g_*\mathbf{N}$ is the pushforward measure of $\mathbf{N}$ by the function

$$g(t,z) := (\tau t, z/\tau).$$

Note that $\tilde{\mathbf{N}}(du,dz)$ is again a Poisson measure of intensity $dudz$, and so $(Y^{\boldsymbol{d},\nu}_{\tau t,\tau s})$ is a (weak) solution of (3) for $\tilde{f} := \tau f$, $\tilde{b} := \tau b$ and $\tilde{\boldsymbol{a}} := \tau\boldsymbol{a}$. So, in particular (taking $\nu = \delta_0$), if we define:

$$\begin{aligned}&\frac{d}{dt}\varphi^{\boldsymbol{a},\tau}_{t,s}(0) = \tau b(\varphi^{\boldsymbol{a},\tau}_{t,s}(0)) + \tau a(t); \quad \varphi^{\boldsymbol{a},\tau}_{s,s}(0) = 0, \\ &H_{\boldsymbol{a},\tau}(t,s) := \exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right), \\ &K_{\boldsymbol{a},\tau}(t,s) := \tau f(\varphi^{\boldsymbol{a},\tau}_{t,s}(0))\exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right),\end{aligned} \tag{49}$$

we have

Lemma 25. *Let $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi}$. Set, for all $t \in \mathbb{R}$, $d(t) := a(\frac{t}{\tau})$. Then it holds that*

$$\forall t \geq s, \quad H_{\boldsymbol{a},\tau}(t,s) = H_{\boldsymbol{d}}(\tau t, \tau s) \quad \text{and} \quad K_{\boldsymbol{a},\tau}(t,s) = \tau K_{\boldsymbol{d}}(\tau t, \tau s).$$

In view of this result, we deduce that $\rho_{\boldsymbol{a},\tau}$ solves

$$\rho_{\boldsymbol{a},\tau}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \quad 1 = \tau \int_{-\infty}^{t} H_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \tag{50}$$

or equivalently, setting

$$\forall t,s \in [0,2\pi], \quad K^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k\geq 0} K_{\boldsymbol{a},\tau}(t, s-2\pi k) \quad \text{and} \quad H^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k \geq 0} H_{\boldsymbol{a},\tau}(t, s-2\pi k), \tag{51}$$

one has, using the same operator notation as in (44)

$$\rho_{\boldsymbol{a},\tau} = K^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}), \quad 1 = \tau H^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}).$$

Note that $\rho_{\cdot,\tau}$ and $\rho_{\cdot}$ are linked by (25). Consequently equations (50) define a unique 2π-periodic continuous function

$$\rho_{\boldsymbol{a},\tau} = \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}, \tag{52}$$

where $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain with transition probability kernel $K^{2\pi}_{\boldsymbol{a},\tau}$ and $c_{\boldsymbol{a},\tau}$ is the constant given by

$$c_{\boldsymbol{a},\tau} := \tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

3.5 Regularity of ρ

The goal of this section is to study the regularity of $\rho_{\boldsymbol{a},\tau}$ with respect to $\boldsymbol{a}$ and τ. For $\eta_0 > 0$, recall that $B^{2\pi}_{\eta_0}$ is the open ball of $C^0_{2\pi}$ defined by (33). The main result of this section is

Proposition 26. *Grant Assumptions 2, 3 and let $\alpha_0 > 0$ such that Assumption 4 holds. Let $\tau_0 > 0$. There exists $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a}, \tau) & \mapsto & \rho_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

The proof of Proposition 26 relies on (52) and on Lemma 29 below, which states that the function $(\boldsymbol{a}, \tau) \mapsto \pi_{\boldsymbol{a},\tau}$ is $\mathcal{C}^2$. We set:

$$C^{0,0}_{2\pi} := \{u \in C^0_{2\pi} | \int_0^{2\pi} u(s)ds = 0\}.$$

Let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}$ and $\tau > 0$. Because $\int_0^{2\pi} \pi_{\boldsymbol{a},\tau}(u)du = 1$, the space $C^0_{2\pi}$ can be decomposed in the following way

$$C^0_{2\pi} = \mathrm{Span}(\pi_{\boldsymbol{a},\tau}) \oplus C^{0,0}_{2\pi}.$$

We denote by $K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}$ the restriction of $K^{2\pi}_{\boldsymbol{a},\tau}$ to $C^0_{2\pi}$ (recall that the linear operator $h \mapsto K^{2\pi}_{\boldsymbol{a},\tau} h$ it defined for all $h \in L^1([0,2\pi])$). Similarly, we denote by $I|_{C^0_{2\pi}}$ the identity operator on $C^0_{2\pi}$. Given a linear operator L, we denote by $N(L)$ its kernel (null-space) and by $R(L)$ its range.

Lemma 27. *Grant Assumptions 2 and 3, let $\alpha_0 > 0$ such that Assumption 4 holds and let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)$, where $\eta_0 > 0$ is given by Lemma 18. It holds that*

$$N(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}) = \mathrm{Span}(\pi_{\boldsymbol{a},\tau}) \quad \text{and} \quad R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}.$$

Proof. We proved in Lemma 20 that $N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \mathrm{Span}(\pi_{\boldsymbol{a},\tau})$. It remains to show that $R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}$. By the Fredholm alternative, we have

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = N(I - (K^{2\pi}_{\boldsymbol{a},\tau})')^{\perp},$$

where $(K^{2\pi}_{\boldsymbol{a},\tau})' \in \mathcal{L}\left(L^\infty([0,2\pi]), L^\infty([0,2\pi])\right)$ is the dual operator of $K^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}\left(L^1([0,2\pi]), L^1([0,2\pi])\right)$. In the proof of Lemma 20, it is shown that

$$\mathbf{1} \in N(I - (K^{2\pi}_{\boldsymbol{a},\tau})'),$$

where $\mathbf{1}$ denotes the constant function equal to 1 on $[0,2\pi]$. The Fredholm alternative yields

$$\dim N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \dim N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = 1.$$

So

$$N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \mathrm{Span}(\mathbf{1}).$$

It follows that

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \mathrm{Span}(\mathbf{1})^{\perp} = \{u \in L^1([0,2\pi]) | \int_0^{2\pi} u(s)ds = 0\}.$$

Finally, using that for $h \in L^1([0,2\pi])$, one has $K^{2\pi}_{\boldsymbol{a},\tau} h \in C^0_{2\pi}$, one obtains the result for the restrictions to $C^0_{2\pi}$. $\square$

As a consequence, the linear operator $I - K^{2\pi}_{\boldsymbol{a},\tau} : C^{0,0}_{2\pi} \to C^{0,0}_{2\pi}$ is inversible, with a continuous inverse.

Lemma 28. *Grant Assumptions 2, 3 and let $\alpha_0 > 0$ such that Assumption 4 holds. Let $\tau_0 > 0$. There exists $\eta_0, \epsilon_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the following function is $\mathcal{C}^2$ Fréchet differentiable*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & \mathcal{L}(C^0_{2\pi}; C^0_{2\pi}) \\ (\boldsymbol{a}, \tau) & \mapsto & H^{2\pi}_{\boldsymbol{a},\tau}. \end{array}$$

The same result holds for $K^{2\pi}_{\boldsymbol{a},\tau}$.

Proof. We only prove the result for H, the proof for K being similar. Let $\epsilon_0 > 0$ be chosen arbitrary such that $\epsilon_0 < \tau_0$.
Step 1. We introduce relevant Banach spaces: E denotes the set of continuous functions

$$E := \mathcal{C}([0,2\pi]^2; \mathbb{R}), \quad \text{equipped with } ||w||_E := \sup_{t,s} |w(t,s)|$$

$$E_0 := \{w \in E,\ \forall s \in [0,2\pi],\ w(2\pi, s) = w(0,s)\}, \quad \text{equipped with } ||\cdot||_E.$$

We define the following application Φ,

$$\begin{array}{rcl} E_0 & \to & \mathcal{L}(C^0_{2\pi}; C^0_{2\pi}) \\ w & \mapsto & \Phi(w) := \left[h \mapsto \left(\int_0^{2\pi} w(t,s)h(s)ds \right)_{t \in [0,2\pi]} \right]. \end{array}$$

Note that Φ is linear and continuous, so in particular $\mathcal{C}^2$. So, to prove the result, it suffices to show that

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E_0 \\ (\boldsymbol{a}, \tau) & \mapsto & (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s \in [0,2\pi]^2} \end{array}$$

is $\mathcal{C}^2$, where $H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is explicitly given by the series (51).
Step 2. Let $k \in \mathbb{N}$ be fixed. We prove that the function

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E \\ (\boldsymbol{a}, \tau) & \mapsto & (H_{\boldsymbol{a},\tau}(t, s - 2\pi k))_{t,s} \end{array}$$

is $\mathcal{C}^2$. To proceed, we use the explicit expression of $H_{\boldsymbol{a},\tau}(t,s)$, given by (49). Note that we have first to show that the function $(\boldsymbol{a},\tau) \mapsto \varphi^{\boldsymbol{a},\tau}_{t,s}(0) \in \mathbb{R}$ is $\mathcal{C}^2$. This follows (see [Fle80, Th. 3.10.2]) from the fact that $b : \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{C}^2$ and so the solution of the ODE (49) is $\mathcal{C}^2$ with respect to $\boldsymbol{a}$ and τ. Moreover, we have for all $h \in C^0_{2\pi}$,

$$D_{\boldsymbol{a}} \varphi^{\boldsymbol{a},\tau}_{t,s}(0) \cdot h = \int_s^t \tau h(u) \exp\left(\tau \int_u^t b'(\varphi^{\boldsymbol{a},\tau}_{\theta,s}(0)) d\theta\right) du.$$

A similar expression holds for $\frac{d}{d\tau}\varphi^{\boldsymbol{a},\tau}_{t,s}(0)$. Using that f is $\mathcal{C}^2$, we deduce that the function

$$(\boldsymbol{a},\tau) \mapsto (H_{\boldsymbol{a},\tau}(t, s - 2\pi k))_{t,s} \in E$$

is $\mathcal{C}^2$ Furthermore, we have for instance

$$D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t,s) \cdot h = -H_{\boldsymbol{a},\tau}(t,s) \int_s^t \tau f'(\varphi^{\boldsymbol{a},\tau}_{u,s}(0)) \left[D_{\boldsymbol{a}} \varphi^{\boldsymbol{a},\tau}_{u,s} \cdot h\right] du.$$

So, proceeding as in the proof of Lemma 18, we deduce the existence of $\eta_0, \lambda_0, A_0 > 0$ (only depending on b, f, α_0, τ_0 and ϵ_0) such that for all $h \in C^0_{2\pi}$ and for all $\tau \in (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$, it holds that

$$\sup_{t,s \in [0,2\pi]^2} \sup_{\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)} |D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s - 2\pi k) \cdot h| \leq A_0 ||h||_\infty e^{-2\pi k \lambda_0}.$$

Similar estimates hold for the second derivative with respect to $\boldsymbol{a}$ and for the first and second derivative with respect to τ.

Step 3. To prove that $(\boldsymbol{a},\tau) \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s} \in E$ is $\mathcal{C}^2$, we rely on [Car67, Th. 3.6.1]. We have

$$\sum_{k \geq 0} \sup_{t,s \in [0,2\pi]^2} \sup_{\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)} \sup_{h \in C^0_{2\pi}, ||h||_\infty \leq 1} |D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s - 2\pi k) \cdot h| \leq \sum_{k \geq 0} A_0 e^{-2\pi k \lambda_0} < \infty.$$

This proves that $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s}$ is Fréchet differentiable, with for all $h \in C^0_{2\pi}$

$$D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau}(t,s) \cdot h = \sum_{k \geq 0} D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s - 2\pi k) \cdot h.$$

Note that this last series converges again normally, and so $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s}$ is in fact $\mathcal{C}^1$. Applying again [Car67, Th. 3.6.1], we prove the same way that $\boldsymbol{a} \mapsto H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is $\mathcal{C}^2$. The same arguments shows that $\tau \mapsto H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is $\mathcal{C}^2$.

Step 4. It remains to prove that $(\boldsymbol{a},\tau) \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s} \in E_0$ is $\mathcal{C}^2$ (we have proved the result for E, not E_0, in the previous step). Let $t, s \in [0, 2\pi]$ be fixed, define

$$w \in E, \quad \mathcal{E}^t_s(w) := w(t,s) \in \mathbb{R}$$

The application $\mathcal{E}^t_s$ is linear and continuous. Moreover, we have seen that $H^{2\pi}_{\boldsymbol{a},\tau} \in E_0$, so

$$\forall s \in [0, 2\pi], \quad \mathcal{E}^{2\pi}_s(H^{2\pi}_{\boldsymbol{a},\tau}) = \mathcal{E}^0_s(H^{2\pi}_{\boldsymbol{a},\tau}).$$

Differentiating with respect to $\boldsymbol{a}$, we deduce that for all $h \in C^0_{2\pi}$,

$$\forall s \in [0, 2\pi], \quad \mathcal{E}^{2\pi}_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \cdot h) = \mathcal{E}^0_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \cdot h),$$

and so $D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}(C^0_{2\pi}, E_0)$. The same results holds for the second derivative with respect to $\boldsymbol{a}$ and the two derivatives with respect to τ. This ends the proof. $\square$

Lemma 29. *Grant Assumptions 2, 3 and let $\alpha_0 > 0$ such that Assumption 4 holds. Let $\tau_0 > 0$. There exists $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a}, \tau) & \mapsto & \pi_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

Remark 30. *Recall that $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain having $K^{2\pi}_{\boldsymbol{a},\tau}$ has kernel transition probability. So, we study the smoothness of the invariant measure with respect to the parameters $(\boldsymbol{a},\tau)$, knowing the smoothness of the transition probability kernel $(\boldsymbol{a},\tau)\mapsto K^{2\pi}_{\boldsymbol{a},\tau}$. We refer to [GM86] for such sensibility result in the setting of finite discrete-time Markov Chains. Our approach is different and based on the Implicit Function Theorem.*

Proof. Let α_0 and τ_0 be fixed. Let $\delta_0,\epsilon_0>0$ be given by Lemma 28. Consider the following $\mathcal{C}^2$-Fréchet differentiable function:

$$\begin{array}{rrcl} F: & C^{0,0}_{2\pi}\times B^{2\pi}_{\eta_0}(\alpha_0)\times(\tau_0-\epsilon_0,\tau_0+\epsilon_0) & \to & C^{0,0}_{2\pi} \\ & (h,\boldsymbol{a},\tau) & \mapsto & (\alpha_0+h)-K^{2\pi}_{\boldsymbol{a},\tau}(\alpha_0+h). \end{array}$$

It holds that $F(0,\alpha_0,\tau_0)=0$. Moreover

$$D_hF(0,\alpha_0,\tau_0)=I-K^{2\pi}_{\alpha_0,\tau_0}\in\mathcal{L}(C^{0,0}_{2\pi},C^{0,0}_{2\pi}),$$

which is invertible with continuous inverse by Lemma 27. So the Implicit Function Theorem applies: there exists $(V^{0,0}_{2\pi},\,V^0_{2\pi},\,V_{\tau_0})$ open neighborhoods of $(0,\alpha_0,\tau_0)$ in $C^{0,0}_{2\pi}\times C^0_{2\pi}\times\mathbb{R}^*_+$ and a $\mathcal{C}^2$-Fréchet differentiable function $U:V^0_{2\pi}\times V_{\tau_0}\to V^{0,0}_{2\pi}$ such that

$$\forall h,\boldsymbol{a},\tau\in V^{0,0}_{2\pi}\times V^0_{2\pi}\times V_{\tau_0},\quad F(h,\boldsymbol{a},\tau)=0\iff h=U(\boldsymbol{a},\tau).$$

By uniqueness of the invariant measure of the Markov chain with transition kernel $K^{2\pi}_{\boldsymbol{a},\tau}$, we deduce that

$$\pi_{\boldsymbol{a},\tau}=\alpha_0+U(\boldsymbol{a},\tau),$$

which is a $\mathcal{C}^2$-Fréchet differentiable function of $(\boldsymbol{a},\tau)$. $\square$

Proof of Proposition 26. Recall that $\rho_{\boldsymbol{a},\tau}=\frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}$, where the constant $c_{\boldsymbol{a},\tau}$ is given by

$$c_{\boldsymbol{a},\tau}=\tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

Furthermore, it holds that $\pi_{\alpha,\tau}=\frac{1}{2\pi}$ and $\rho_{\alpha,\tau}=\gamma(\alpha)$ (see (48)). So $c_{\alpha,\tau}=\frac{1}{2\pi\gamma(\alpha)}>0$. So for ϵ_0,η_0 small enough, it holds that

$$\forall\boldsymbol{a}\in B^{2\pi}_{\eta_0}(\alpha_0),\forall\tau\in(\tau_0-\epsilon_0,\tau_0+\epsilon_0),\quad c_{\boldsymbol{a},\tau}>0.$$

So, using Lemmas 28 and 29, it holds that c and ρ are $\mathcal{C}^2$, which ends the proof. $\square$

As a first application of this result, we prove that the mean number of spikes of a neuron driven by a periodic input only depends on the mean of the input current.

Proposition 31. *Grant Assumptions 2, 3 and let $\alpha_0>0$ such that Assumption 4 holds. Let $\tau_0>0$ and consider η_0 be given by Proposition 26. Let $h\in C^{0,0}_{2\pi}$ such that $\alpha_0+h\in B^{2\pi}_{\eta_0}(\alpha_0)$. It holds that*

$$c_{\alpha_0+h,\tau_0}=c_{\alpha_0,\tau_0}=\frac{1}{2\pi\gamma(\alpha_0)}.$$

We denote by c_{α_0} this last quantity. In particular, the mean number of spikes per period:

$$\frac{1}{2\pi}\int_0^{2\pi}\rho_{\alpha_0+h,\tau_0}(u)du=\gamma(\alpha_0),$$

only depends on α_0 (which is the mean of the external current $(\alpha_0+h(t))_{t\in[0,2\pi]}$).

Proof. Let $\boldsymbol{a}\in B^{2\pi}_{\eta_0}(\alpha_0)$. We prove that

$$\forall h\in C^{0,0}_{2\pi},\quad D_{\boldsymbol{a}}c_{\boldsymbol{a},\tau_0}\cdot h=0.$$

We have $c_{\boldsymbol{a},\tau_0} = \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0}(\pi_{\boldsymbol{a},\tau_0})$. Differentiating with respect to $\boldsymbol{a}$, one gets

$$D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = \tau_0 \left[D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0} D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h.$$

Recall that $\pi_{\boldsymbol{a},\tau_0} = K^{2\pi}_{\boldsymbol{a},\tau_0} \pi_{\boldsymbol{a},\tau_0}$ so

$$D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h = \left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right] \pi_{\boldsymbol{a},\tau_0} + K^{2\pi}_{\boldsymbol{a},\tau_0} \left[D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h\right].$$

So, using Lemma 27, one has

$$D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h = \left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1} \left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right] \pi_{\boldsymbol{a},\tau_0}. \tag{53}$$

Define on $C^{0,0}_{2\pi}$ the linear operator

$$\forall h \in C^{0,0}_{2\pi}, \quad \mathbb{1}^{2\pi}(h)(t) := \int_0^{2\pi} \mathbb{1}_{\{t \geq s\}} h(s) ds = \int_0^t h(s) ds.$$

We have

$$1 * K_{\boldsymbol{a},\tau_0} = 1 - H_{\boldsymbol{a},\tau_0}, \tag{54}$$

so on $C^{0,0}_{2\pi}$,

$$H^{2\pi}_{\boldsymbol{a},\tau_0} = \mathbb{1}^{2\pi}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]. \tag{55}$$

So

$$H^{2\pi}_{\boldsymbol{a},\tau_0}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1} = \mathbb{1}^{2\pi}.$$

Consequently, we have

$$D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = \tau_0 \left[D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0 \mathbb{1}^{2\pi} \left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right] \pi_{\boldsymbol{a},\tau_0}$$

Differentiating (55), one has

$$D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h = -\mathbb{1}^{2\pi}\left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right],$$

and so for all $h \in C^{0,0}_{2\pi}$, $D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = 0$. Then for all $h \in C^{0,0}_{2\pi}$ such that $\alpha_0 + h \in B^{2\pi}_{\eta_0}(\alpha_0)$, one has

$$c_{\alpha_0+h,\tau_0} - c_{\alpha_0,\tau_0} = \int_0^1 \left[D_{\boldsymbol{a}} c_{\alpha_0+th,\tau_0} \cdot h\right] dt = 0.$$

Finally we have $\pi_{\alpha_0,\tau_0} = \frac{1}{2\pi}$ and, by (48), $\rho_{\alpha_0,\tau_0} = \gamma(\alpha_0)$. By definition (52), we have $c_{\alpha_0,\tau_0} = \frac{\pi_{\alpha_0,\tau_0}}{\rho_{\alpha_0,\tau_0}}$. It ends the proof. □

3.6 Strategy to handle the non-linear equation (1)

Grant Assumptions 2, 3 and let $\alpha_0 > 0$ such that Assumption 4 holds. Let $\tau_0 > 0$ be given by Assumption 8. For $\eta_0, \epsilon_0 > 0$, define $G : B^{2\pi}_{\eta_0}(\alpha_0) \cap C^{0,0}_{2\pi} \times (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) \to C^{0,0}_{2\pi}$ such that

$$G(h, \alpha, \tau) := (\alpha + h) - J(\alpha)\rho_{\alpha+h,\tau}. \tag{56}$$

Using Propositions 26 and 31, we choose η_0, ϵ_0 small enough such that G is $\mathcal{C}^2$-Fréchet differentiable and indeed takes values in $C^{0,0}_{2\pi}$. For any constant $\alpha, \tau > 0$, we have, by (48), $\rho_{\alpha,\tau} = \gamma(\alpha)$. Recalling that $J(\alpha)\gamma(\alpha) = \alpha$, we have

$$\forall (\alpha, \tau) \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0), \quad G(0, \alpha, \tau) = 0. \tag{57}$$

Those are the trivial roots of G. To construct the periodic solutions to (1), we find the non-trivial roots of G. In fact, Theorem 14 is deduced from the following proposition.

Proposition 32. *Consider b, f and $\alpha_0, \tau_0 > 0$ such that Assumptions 2, 3, 4, 8, 9 and 12 hold. Let G be defined by* (56). *There exists $X \times V_{\alpha_0} \times V_{\tau_0}$ an open neighborhood of $(0, \alpha_0, \tau_0)$ in $(C^{0,0}_{2\pi}, ||\cdot||_\infty) \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that:*

1. *There exists a continuous curve $\{(h_v, \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ of real 2π-periodic solutions of* (24) *passing through $(0, \alpha_0, \tau_0)$ at $v = 0$ and such that for all $v \in (-v_0, v_0)$*

$$(h_v, \alpha_v, \tau_v) \in X \times V_{\alpha_0} \times V_{\tau_0} \quad \text{and} \quad G(h_v, \alpha_v, \tau_v) = 0.$$

   Moreover, it holds that

$$\forall v \in (-v_0, v_0), \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\cos(t)dt = v \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\sin(t)dt = 0.$$

   In particular, $h_v \not\equiv 0$ for $v \neq 0$.

2. *For all $(h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0}$, with $h \not\equiv 0$, it holds that*

$$G(h, \alpha, \tau) = 0 \iff [\exists v \in (-v_0, v_0), \exists \theta \in [0, 2\pi), \quad (h, \alpha, \tau) \equiv (S_\theta h_v, \alpha_v, \tau_v)].$$

We here prove that our main result is a consequence of this proposition.

Proof that Proposition 32 implies Theorem 14. Let (h_v, α_v, τ_v) be the continuous curve given by Proposition 32. Define $\boldsymbol{a}_v$

$$\forall t \in \mathbb{R}, \quad a_v(t) := \alpha_v + h_v(t/\tau_v).$$

The function $\boldsymbol{a}_v$ is $2\pi\tau_v$-periodic and continuous. From $G(h_v, \alpha_v, \tau_v) = 0$, we deduce that

$$\boldsymbol{a}_v = J(\alpha_v)\rho_{\boldsymbol{a}_v}.$$

Consider $\tilde{\nu}_{\boldsymbol{a}_v}$ defined by (46). By Proposition 24, $(\tilde{\nu}_{\boldsymbol{a}_v}(t))$ is a $2\pi\tau_v$-periodic solution of (1) and $(\tilde{\nu}_{\boldsymbol{a}_v}, \alpha_v, \tau_v)$ satisfies all the properties stated in Theorem 14: this gives the existence part of the proof. We now prove uniqueness.

Let $\epsilon_0 > 0$ small enough such that $(\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) \subset V_{\tau_0}$, V_{τ_0} being given by Proposition 32. Let $J, \tau > 0$ be fixed, consider $\nu(t)$ a $2\pi\tau$-periodic solution of (1) such that

$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t \in [0, 2\pi\tau]} \left| J \int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0 \right| < \epsilon_1,$$

for some constant $\epsilon_1 > 0$ to be specified later. Define $\boldsymbol{a}$

$$\forall t \in \mathbb{R}, \quad a(t) := J \int_{\mathbb{R}_+} f(x)\nu(t, dx).$$

The function $\boldsymbol{a}$ is $2\pi\tau$-periodic. Let $(X_t)_{t \geq 0}$ be the solution of the non-linear equation (1), starting with the initial condition $\nu(0) \in \mathcal{M}(f^2)$. The arguments of [CTV20, Lem. 24] show that, under Assumptions 2 and 3, the function $t \mapsto \mathbb{E} f(X_t)$ is continuous, and so $\boldsymbol{a} \in C^0_{2\pi\tau}$. We write

$$a(t) =: \alpha + h(t/\tau),$$

for some constant α and some $h \in C^{0,0}_{2\pi}$. Because $\nu(t)$ is a periodic solution of (1), it holds that

$$\boldsymbol{a} = J\rho_{\boldsymbol{a}},$$

or equivalently,

$$\alpha + h = J\rho_{\alpha+h,\tau}. \tag{58}$$

We have by assumption

$$|\alpha-\alpha_0| = \left| \frac{1}{2\pi}\int_0^{2\pi} J \int_{\mathbb{R}_+} f(x)\nu(\tau u, dx)du - \frac{1}{2\pi}\int_0^{2\pi} J(\alpha_0)\int_{\mathbb{R}_+} f(x)\nu^\infty_{\alpha_0}(dx)du \right| < \epsilon_1.$$

Recall that α_0 satisfies Assumption 4. By Lemma 17 and using the continuity of b', we can assume that ϵ_1 is small enough such that Assumption 4 is also satisfied by α. Let η_0 be given by Proposition 26 (η_0 only depends on b, f, α_0 and τ_0). Provided that $\epsilon_1 \leq \eta_0$, we can apply Proposition 31 at (α,τ). It holds that

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha+h,\tau}(u)du = \gamma(\alpha),$$

so

$$\alpha = J\gamma(\alpha).$$

This proves that $J = J(\alpha)$. So (58) implies that $G(h,\alpha,\tau) = 0$. By the uniqueness part of Proposition 32, there exists $\theta \in [0, 2\pi)$ and $v \in (-v_0, v_0)$ such that

$$\forall t, \quad h(t) = h_v(t+\theta), \quad \alpha = \alpha_v, \quad \tau = \tau_v.$$

So, we deduce that $a(t) = \alpha_v + h_v\left(\frac{t+\theta}{\tau_v}\right)$ and $J = J(\alpha_v)$. This ends the proof. $\square$

It remains to prove Proposition 32.

3.7 Linearization of G.

We consider for all $\alpha, \tau \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$:

$$B_{\alpha,\tau} := D_h G(0,\alpha,\tau).$$

Define:

$$\forall t \in \mathbb{R}, \quad \Theta_{\alpha,\tau}(t) := \tau\Theta_\alpha(\tau t)\mathbb{1}_{\{t\geq 0\}}, \tag{59}$$

where Θ_α is given by (14). The main result of this section is the following.

Proposition 33. *Let $h \in C^{0,0}_{2\pi}$. It holds that*

$$B_{\alpha,\tau}(h)(t) = h(t) - J(\alpha)\int_{\mathbb{R}} \Theta_{\alpha,\tau}(t-s)h(s)ds.$$

The proof of this proposition relies on Lemmas 34 and 35 below. Let $h \in C^{0,0}_{2\pi}$, it holds that

$$B_{\alpha,\tau}(h) = h - J(\alpha)D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h.$$

By equation (52) and Proposition 31, one has

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h = \frac{1}{c_\alpha}D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h.$$

To compute $D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h$, we use (53) with $\boldsymbol{a} \equiv \alpha$:

$$D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h = (I - K^{2\pi}_{\alpha,\tau})^{-1}\left[D_{\boldsymbol{a}}K^{2\pi}_{\alpha,\tau}\cdot h\right](\tfrac{1}{2\pi}).$$

The next lemma is devoted to the computation of $(I-K^{2\pi}_{\alpha,\tau})^{-1}$. Consider $t \mapsto r_\alpha(t)$ the solution of the convolution Volterra integral equation (6) (with $\nu = \delta_0$ and $\boldsymbol{a} = \alpha$). That is, r_α solves $r_\alpha = K_\alpha + K_\alpha * r_\alpha$. By [CTV20, Prop. 37], there exists a function $\xi_\alpha \in L^1(\mathbb{R}_+)$ such that for all $t \geq 0$,

$$r_\alpha(t) = \gamma(\alpha) + \xi_\alpha(t).$$

Define for all $t \geq 0$, $r_{\alpha,\tau}(t) := \tau r_\alpha(\tau t)$. It solves

$$r_{\alpha,\tau} = K_{\alpha,\tau} + K_{\alpha,\tau} * r_{\alpha,\tau}, \tag{60}$$

where $K_{\alpha,\tau}$ is given by (49). Similarly, let $\xi_{\alpha,\tau}(t) := \tau \xi_\alpha(\tau t)$. We have

$$r_{\alpha,\tau}(t) = \tau\gamma(\alpha) + \xi_{\alpha,\tau}(t).$$

Remind that by definition, we have

$$K^{2\pi}_{\alpha,\tau}(h)(t) = \int_0^{2\pi} K^{2\pi}_{\alpha,\tau}(t,s)h(s)ds = \int_{-\infty}^{t} K_{\alpha,\tau}(t-s)h(s)ds.$$

Lemma 34. *The inverse of the linear operator $I - K^{2\pi}_{\alpha,\tau} : C^{0,0}_{2\pi} \to C^{0,0}_{2\pi}$ is given by $I + r^{2\pi}_{\alpha,\tau}$ where for all $h \in C^{0,0}_{2\pi}$ and $t \in [0, 2\pi]$*

$$\begin{aligned}
r^{2\pi}_{\alpha,\tau}(h) &:= \tau\gamma(\alpha)\Gamma(h) + \xi^{2\pi}_{\alpha,\tau}(h),\\
\Gamma(h)(t) &:= \int_0^t h(s)ds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds,\\
\xi^{2\pi}_{\alpha,\tau}(h)(t) &:= \int_{-\infty}^{t} \xi_{\alpha,\tau}(t-s)h(s)ds.
\end{aligned}$$

Proof. Note that $\Gamma(h)$ is the only primitive of h which belongs to $C^{0,0}_{2\pi}$. Moreover, because $t \mapsto \xi_{\alpha,\tau}(t) \in L^1(\mathbb{R}_+)$, we have for $h \in C^{0,0}_{2\pi}$:

$$\int_0^{2\pi}\int_{-\infty}^{t} \xi_{\alpha,\tau}(t-s)h(s)dsdt = \int_0^{2\pi}\int_0^{\infty} \xi_{\alpha,\tau}(u)h(t-u)dudt = \int_0^{\infty}\int_0^{2\pi} \xi_{\alpha,\tau}(u)h(t-u)dtdu = 0.$$

So, $\xi^{2\pi}_{\alpha,\tau}(h) \in C^{0,0}_{2\pi}$ and so $r^{2\pi}_{\alpha,\tau}$ is well-defined. To conclude, we have to show that on $C^{0,0}_{2\pi}$

$$K^{2\pi}_{\alpha,\tau} \circ r^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau} \circ K^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau} - K^{2\pi}_{\alpha,\tau}.$$

Note that for all $t \in [0, 2\pi]$,

$$\frac{d}{dt}\left[\Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t)\right] = K^{2\pi}_{\alpha,\tau}(h)(t).$$

Because $\Gamma(h), H^{2\pi}_{\alpha,\tau}(h) \in C^{0,0}_{2\pi}$, we deduce that

$$\Gamma(K^{2\pi}_{\alpha,\tau}(h)) = \Gamma(h) - H^{2\pi}_{\alpha,\tau}.$$

Moreover, we have (using that $\xi_{\alpha,\tau}, K_{\alpha,\tau} \in L^1(\mathbb{R}_+)$)

$$\begin{aligned}
\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))(t) &= \int_{-\infty}^{t} \xi_{\alpha,\tau}(t-s)\int_{-\infty}^{s} K_{\alpha,\tau}(s-u)h(u)duds\\
&= \int_{-\infty}^{t} h(u)\int_u^t \xi_{\alpha,\tau}(t-s)K_{\alpha,\tau}(s-u)dsdu\\
&= \int_{-\infty}^{t} h(u)(\xi_{\alpha,\tau} * K_{\alpha,\tau})(t-u)du.
\end{aligned}$$

Using (54) and (60), we deduce the identity

$$K_{\alpha,\tau} * \xi_{\alpha,\tau} = \xi_{\alpha,\tau} * K_{\alpha,\tau} = \xi_{\alpha,\tau} - K_{\alpha,\tau} + \tau\gamma(\alpha)H_{\alpha,\tau}. \tag{61}$$

So

$$\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h) + \tau\gamma(\alpha)H^{2\pi}_{\alpha,\tau}(h).$$

Altogether,

$$r^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h).$$

We now prove that $K^{2\pi}_{\alpha,\tau}(r^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h)$. Using (61), we have $K^{2\pi}_{\alpha,\tau}(\xi^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))$. Moreover, because $K^{2\pi}_{\alpha,\tau}(1) = 1$, we have

$$\begin{aligned}
K^{2\pi}_{\alpha,\tau}(\Gamma(h))(t) &= \int_{-\infty}^{t} K_{\alpha,\tau}(t-s)\int_0^s h(u)duds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds \\
&= \left[H_{\alpha,\tau}(t-s)\int_0^s h(u)du\right]_{-\infty}^{t} - \int_{-\infty}^{t} H_{\alpha,\tau}(t-s)h(s)ds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds \\
&= \Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t) = \Gamma(K^{2\pi}_{\alpha,\tau}(h))(t).
\end{aligned}$$

It ends the proof. $\square$

So for all $\alpha, \tau \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$ and $h \in C^{0,0}_{2\pi}$, it holds that

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h = \frac{1}{c_{\alpha,\tau}} D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h = (I + r^{2\pi}_{\alpha,\tau})\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau}\cdot h\right](\gamma(\alpha)). \tag{62}$$

Define for all $t \geq 0$, $\Xi_{\alpha,\tau}(t) := \tau\Xi_\alpha(\tau t)$ and denote by $\Xi^{2\pi}_{\alpha,\tau}$ the linear operator

$$\forall h \in C^0_{2\pi}, \forall t \in [0, 2\pi], \quad \Xi^{2\pi}_{\alpha,\tau}(h)(t) := \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u)h(u)du.$$

Lemma 35. *For all $h \in C^0_{2\pi}$ we have $\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau}\cdot h\right](\gamma(\alpha)) = \Xi^{2\pi}_{\alpha,\tau}(h)$.*

Proof. Given $h \in C^0_{2\pi}$, we have

$$\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau}\cdot h\right](\gamma(\alpha))(t) = \gamma(\alpha)\int_{-\infty}^{t}\left[D_{\boldsymbol{a}} K_{\alpha,\tau}\cdot h\right](t,s)ds$$

So we have to prove that

$$\forall h \in C^0_{2\pi}, \quad \gamma(\alpha)\int_{-\infty}^{t}\left[D_{\boldsymbol{a}} K_{\alpha,\tau}\cdot h\right](t,s)ds = \int_{-\infty}^{t}\Xi_{\alpha,\tau}(t-s)h(s)ds.$$

When $\tau = 1$, this computation is done in [Cor20, Lem. 61]. It is first proved that

$$\gamma(\alpha)\int_{-\infty}^{t}\left[D_{\boldsymbol{a}} H_\alpha\cdot h\right](t,s)ds = -\int_{\mathbb{R}}\Psi_\alpha(t-s)h(s)ds,$$

where $\Psi_\alpha(t)$ is given by (19). This computation relies on the explicit expression satisfied by $\left[D_{\boldsymbol{a}} H_\alpha\cdot h\right](t,s)$, namely

$$\left[D_{\boldsymbol{a}} H_\alpha\cdot h\right](t,s) = H_\alpha(t-s)\int_s^t f'(\varphi^\alpha_{u-s}(0))\int_s^u h(\theta)\exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0))dv\right)d\theta du.$$

Using Fubini's Theorem and the identity

$$\int_\theta^t f'(\varphi^\alpha_{u-s}(0))\exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0))dv\right)du = \frac{f(\varphi^\alpha_{t-s}(0)) - f(\varphi^\alpha_{\theta-s}(0))}{b(\varphi^\alpha_{\theta-s}(0)) + \alpha}$$

lead to the convolution between Ψ_α and h. We refer to [Cor20, Lem. 61] for more details. Then one uses that

$$\int_{-\infty}^{t}\left[D_{\boldsymbol{a}} K_\alpha\cdot h\right](t,s)ds = -\frac{d}{dt}\int_{-\infty}^{t}\left[D_{\boldsymbol{a}} H_\alpha\cdot h\right](t,s)ds$$

and that $\Xi_\alpha(t) = \frac{d}{dt}\Psi_\alpha(t)$ to obtain the stated identity with $\tau = 1$. The result for $\tau \neq 1$ can be deduced from the case $\tau = 1$. Indeed, given $\alpha > 0$ and $h \in C^0_{2\pi}$, define $\tilde{f} := \tau f$, $\tilde{b} := \tau b$, $\tilde{\alpha} := \tau\alpha$, and $\tilde{h} := \tau h$. By applying the result for $\tilde{\tau} := 1$, $\tilde{b}$, $\tilde{f}$, $\tilde{\alpha}$ and $\tilde{h}$, we obtain exactly the stated equality. $\square$

Proof of Proposition 33. We use Lemma 35 together with (62). For all $h \in C^{0,0}_{2\pi}$, one obtains

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h = \Xi^{2\pi}_{\alpha,\tau}(h) + r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

The definition of $r^{2\pi}_{\alpha,\tau}$ yields

$$r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)) = \tau\gamma(\alpha)\Gamma(\Xi^{2\pi}_{\alpha,\tau}(h)) + \xi^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

Let $\Psi_{\alpha,\tau}(t) := \Psi_\alpha(\tau t)$, such that $\frac{d}{dt}\Psi_{\alpha,\tau}(t) = \Xi_{\alpha,\tau}(t)$. From the identity

$$\frac{d}{dt}\int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u)h(u)du = \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u)h(u)du,$$

we find that

$$\Gamma(\Xi^{2\pi}_{\alpha,\tau}(h))(t) = \int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u)h(u)du = \int_{-\infty}^{t} (1 * \Xi_{\alpha,\tau})(t-u)h(u)du.$$

So

$$\begin{aligned}\left[D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h\right](t) = &\int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u)h(u)du + \tau\gamma(\alpha)\int_{-\infty}^{t} (1 * \Xi_{\alpha,\tau})(t-u)h(u)du \\ &+ \int_{-\infty}^{t} \xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u} \Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta\end{aligned}$$

Fubini's Theorem yields

$$\int_{-\infty}^{t} \xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u} \Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta = \int_{-\infty}^{t} (\xi_{\alpha,\tau} * \Xi_{\alpha,\tau})(t-\theta)h(\theta)d\theta.$$

Finally, we have

$$\begin{aligned}\Xi_{\alpha,\tau} + \tau\gamma(\alpha)(1 * \Xi_{\alpha,\tau}) + \xi_{\alpha,\tau} * \Xi_{\alpha,\tau} &= \Xi_{\alpha,\tau} + r_{\alpha,\tau} * \Xi_{\alpha,\tau} \quad \text{(because } r_{\alpha,\tau} = \tau\gamma(\alpha) + \xi_{\alpha,\tau}\text{)} \\ &\overset{(21)}{=} \Theta_{\alpha,\tau},\end{aligned}$$

so

$$\left[D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h\right](t) = \int_{-\infty}^{t} \Theta_{\alpha,\tau}(t-u)h(u)du.$$

It ends the proof. $\square$

3.8 The linearization of G at $(0, \alpha_0, \tau_0)$ is a Fredholm operator

For notational convenience we now write

$$B_0 := B_{\alpha_0,\tau_0} = D_h G(0, \alpha_0, \tau_0).$$

Proposition 36. *We have $N(B_0) = R(Q)$, $R(B_0) = N(Q)$, where Q is the following projector on $C^{0,0}_{2\pi}$:*

$$\forall z \in C^{0,0}_{2\pi}, \quad Q(z)(t) := \left[\frac{1}{2\pi}\int_0^{2\pi} z(s)e^{-is}ds\right]e^{it} + \left[\frac{1}{2\pi}\int_0^{2\pi} z(s)e^{is}ds\right]e^{-it}. \tag{63}$$

Remark 37. *In particular, $B_0 \in \mathcal{L}(C^{0,0}_{2\pi}, C^{0,0}_{2\pi})$ is a Fredholm operator of index 0, with dim $N(B_0) =$ 2.*

Proof. First, let $h \in N(B_0)$. One has for all $t \in \mathbb{R}$

$$h(t) = J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds.$$

Consider for all $n \in \mathbb{Z}$

$$\tilde{h}_n := \frac{1}{2\pi} \int_0^{2\pi} h(s)e^{-ins}ds$$

the n-th Fourier coefficient of h. We have

$$\forall n \in \mathbb{Z}, \quad \tilde{h}_n = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)\tilde{h}_n.$$

Assumption 9 ensures that

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in) \neq 1,$$

and so

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad \tilde{h}_n = 0.$$

We deduce that $h \in R(Q)$. Conversely, if $h \in R(Q)$, there exists $c \in \mathbb{C}$ such that

$$h(t) = ce^{it} + \bar{c}e^{-it}$$

and so

$$\begin{aligned} J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds &= ce^{it}J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(s)e^{-is}ds + \bar{c}e^{-it}J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(s)e^{is}ds \\ &= ce^{it}J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) + \bar{c}e^{-it}J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) \\ &= h(t). \end{aligned}$$

We used here that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) = 1$ (Assumption 8). This proves that $N(B_0) = R(Q)$. Consider now $k \in R(B_0)$, there exists $h \in C^0_{2\pi}$ such that $B_0(h) = k$. We have for all $t \in \mathbb{R}$

$$h(t) - J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds = k(t).$$

Using that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = 1$, we deduce that

$$\frac{1}{2\pi} \int_0^{2\pi} k(s)e^{-is}ds = \left[\frac{1}{2\pi} \int_0^{2\pi} h(s)e^{-is}ds\right] (1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i)) = 0.$$

Similarly, $\frac{1}{2\pi}\int_0^{2\pi} k(s)e^{is}ds = 0$ and so $k \in N(Q)$. It remains to show that $N(Q) \subset R(B_0)$. Consider $h \in N(Q)$ and let

$$\tilde{h}_n := \frac{1}{2\pi} \int_0^{2\pi} h(s)e^{-ins}ds$$

be it's n-th Fourier coefficient. We have $\tilde{h}_1 = \tilde{h}_{-1} = 0$. Define

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad \epsilon_n := \frac{J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}{1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}.$$

The function h is continuous, and so h belongs to $L^2([0,2\pi])$. We deduce that

$$\sum_{n \in \mathbb{Z}\backslash\{-1,1\}} |\tilde{h}_n|^2 < \infty.$$

Moreover, because $\Theta_{\alpha_0,\tau_0} \in L^1(\mathbb{R}_+)$, the Riemann-Lebesgue lemma yields the existence of a constant C such that for $n \in \mathbb{Z}$,

$$|n| > 1 \implies |\epsilon_n| \leq \frac{C}{|n|}.$$

We deduce that

$$\sum_{n\in\mathbb{Z}\setminus\{-1,1\}} |n\epsilon_n \tilde{h}_n|^2 < \infty.$$

Consequently, defining

$$\forall t \in \mathbb{R}, \quad w(t) := \sum_{n\in\mathbb{Z}\setminus\{-1,1\}} \epsilon_n \tilde{h}_n e^{int},$$

it holds that $w \in H^1([0, 2\pi])$, and so w is continuous (see for instance [Bre11, Th. 8.2]). Finally, let $k := h + w$. It holds that $k \in C^0_{2\pi}$ and the n-th Fourier coefficient of k is equals to $\frac{\tilde{h}_n}{1-J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}$. We deduce that $B_0(k) = h$. This ends the proof. □

3.9 The Lyapunov-Schmidt reduction method

The problem of finding the roots of G defined by (56) is an infinite dimensional problem. We use the method of Lyapunov-Schmidt to obtain an equivalent problem of finite-dimension - here of dimension 2. The equation $G = 0$ is equivalent to

$$\begin{aligned} QG(Qh + (I - Q)h, \alpha, \tau) = 0 \\ (I - Q)G(Qh + (I - Q)h, \alpha, \tau) = 0, \end{aligned}$$

where the projector Q is defined by (63). Define the following function W:

$$\begin{array}{rrcl} W: & U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0} & \to & R(B_0) \\ & (v, w, \alpha, \tau) & \mapsto & (I - Q)G(v + w, \alpha, \tau), \end{array}$$

where $U_2 \times W_2$ are open neighborhood of $(0, 0)$ in $N(B_0) \times R(B_0)$.

We have $W(0, 0, \alpha_0, \tau_0) = 0$ and $D_w W(0, 0, \alpha_0, \tau_0) = (I - Q)D_h G(0, \alpha_0, \tau_0) = (I - Q)B_0 \in \mathcal{L}(R(B_0), R(B_0))$ which is bijective with continuous inverse. The Implicit Theorem Function applies: there exists a $\mathcal{C}^1$ function $\psi : N(B_0) \times V_{\alpha_0} \times V_{\tau_0} \mapsto R(B_0)$ such that

$W(v, w, \alpha, \tau) = 0$ for $(v, w, \alpha, \tau) \in U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0}$ is equivalent to
$w = \psi(v, \alpha, \tau)$.

Again, the neighborhoods $U_2, W_2, V_{\tau_0}, V_{\alpha_0}$ may be shrunk in this construction. We deduce that

$$G(h, \alpha, \tau) = 0 \text{ for } (h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \text{ is equivalent to} \tag{64}$$

$$QG(Qh + \psi(Qh, \alpha, \tau), \alpha, \tau) = 0. \tag{65}$$

Note that for all $\theta \in \mathbb{R}$, we have for all $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi}$, $\rho_{S_\theta \boldsymbol{a},\tau} = S_\theta \rho_{\boldsymbol{a},\tau}$. It follows that

$$G(S_\theta h, \alpha, \tau) = S_\theta G(h, \alpha, \tau).$$

Moreover, it is clear that the projection Q commutes with S_θ (for all $\theta \in \mathbb{R}$, $S_\theta Q = QS_\theta$) and by the local uniqueness of the Implicit Theorem Function, we deduce that

$$\psi(S_\theta v, \alpha, \tau) = S_\theta \psi(v, \alpha, \tau).$$

Using that any element $Qh \in N(B_0)$ can be written

$$Qh = t \mapsto ce^{it} + \bar{c}e^{-it} := ce_0 + \bar{c}\bar{e}_0$$

for some $c \in \mathbb{C}$ and using the definition of Q, we deduce that (64) is equivalent to the complex equation:

$$\hat{\Phi}(c,\alpha,\tau) = 0 \text{ for } (c,\alpha,\tau) \in V_0 \times V_{\alpha_0} \times V_{\tau_0}, \text{ where}$$

$$\hat{\Phi}(c,\alpha,\tau) := \frac{1}{2\pi}\int_0^{2\pi} G(ce_0 + \bar{c}\bar{e}_0 + \psi(ce_0 + \bar{c}\bar{e}_0,\alpha,\tau),\alpha,\tau)_t e^{-it} dt$$

and V_0 is an open neighborhood of 0 in $\mathbb{C}$. We have moreover

$$\forall \theta \in \mathbb{R}, \quad \hat{\Phi}(ce^{i\theta},\alpha,\tau) = e^{i\theta}\hat{\Phi}(c,\alpha,\tau),$$

and so (64) is equivalent to

$$\hat{\Phi}(v,\alpha,\tau) = 0 \text{ for } v \in (-v_0, v_0).$$

Note that $\hat{\Phi}(-v,\alpha,\tau) = -\hat{\Phi}(v,\alpha,\tau)$ and in particular

$$\forall \alpha,\tau \in V_{\alpha_0} \times V_{\tau_0}, \quad \hat{\Phi}(0,\alpha,\tau) = 0.$$

This is coherent with (57). In order to eliminate these trivial solutions, following [Kie12], we set for $v \in (-v_0, v_0) \setminus \{0\}$:

$$\begin{aligned}\tilde{\Phi}(v,\alpha,\tau) :=& \frac{\hat{\Phi}(v,\alpha,\tau)}{v} \\ =& \int_0^1 D_v\hat{\Phi}(\theta v,\alpha,\tau)d\theta.\end{aligned}$$

To summarize, we have proved that

Lemma 38. *There exists $v_0 > 0$ and open neighborhoods $X \times V_{\alpha_0} \times V_{\tau_0}$ of $(0,\alpha_0,\tau_0)$ in $C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that the problem*

$$G(h,\alpha,\tau) = 0 \text{ for } (h,\alpha,\tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \text{ with } h \neq 0$$

is equivalent to

$$\tilde{\Phi}(v,\alpha,\tau) = 0 \text{ for } (v,\alpha,\tau) \in (-v_0,v_0) \times V_{\alpha_0} \times V_{\tau_0}.$$

The next section is devoted to the study of this reduced problem.

3.10 Study of the reduced 2D-problem

We denote by cos the cosinus function, such that $ve_0 + v\bar{e_0} = 2v\cos$.

Lemma 39. *We have:*

1. $\tilde{\Phi}(0,\alpha_0,\tau_0) = 0$.
2. $D_\tau\tilde{\Phi}(0,\alpha_0,\tau_0) = \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\tau}G(0,\alpha_0,\tau_0)\cdot 2\cos\right]_t e^{-it}dt$.
3. $D_\alpha\tilde{\Phi}(0,\alpha_0,\tau_0) = \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\alpha}G(0,\alpha_0,\tau_0)\cdot 2\cos\right]_t e^{-it}dt$.

Proof. We have $\tilde{\Phi}(0,\alpha_0,\tau_0) = D_v\hat{\Phi}(0,\alpha_0,\tau_0)$ and

$$D_v\hat{\Phi}(0,\alpha_0,\tau_0) = \frac{1}{2\pi}\int_0^{2\pi} D_hG(0,\alpha_0,\tau_0)\cdot\left[2\cos + D_v\psi(0,\alpha_0,\tau_0)\cdot 2\cos\right]_t e^{-it}dt.$$

Moreover, it holds that [see Kie12, Coroll. 1.2.4]

$$D_v\psi(0,\alpha_0,\tau_0)\cdot\cos = 0$$

and $\cos \in N(D_h G(0,\alpha_0,\tau_0))$, so $\tilde{\Phi}(0,\alpha_0,\tau_0)=0$. To prove the second point (the third point is proved similarly), we have $D_\tau \tilde{\Phi}(0,\alpha_0,\tau_0) = D^2_{v\tau}\hat{\Phi}(0,\alpha_0,\tau_0)$. Moreover,

$$
\begin{aligned}
D_\tau \hat{\Phi}(v,\alpha,\tau) =& \frac{1}{2\pi}\int_0^{2\pi} D_\tau G(2v\cos + \psi(2r\cos,\alpha,\tau),\alpha,\tau)_t e^{-it}dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi} \left[D_h G(2r\cos + \psi(2v\cos,\alpha,\tau),\alpha,\tau)\cdot D_\tau\psi(2v\cos,\alpha,\tau)\right]_t e^{-it}dt.
\end{aligned}
$$

So

$$
\begin{aligned}
D^2_{v\tau}\hat{\Phi}(0,\alpha_0,\tau_0) = & \frac{1}{2\pi}\int_0^{2\pi} \left[D^2_{h\tau}G(0,\alpha_0,\tau_0)\cdot(2\cos + D_v\psi(0,\alpha_0,\tau_0)\cdot 2\cos)\right]_t e^{-it}dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi} \left[D_h G(0,\alpha_0,\tau_0)\cdot D^2_{v\tau}\psi(0,\alpha_0,\tau_0)\cdot 2\cos\right]_t e^{-it}dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi} D^2_{hh}G(0,\alpha_0,\tau_0)\cdot[2\cos + D_v\psi(0,\alpha_0,\tau_0)\cdot 2\cos, D_\tau\psi(0,\alpha_0,\tau_0)]_t e^{-it}dt.
\end{aligned}
$$

Note that for all α,τ in the neighborhood of α_0,τ_0, one has

$$\psi(0,\alpha,\tau)=0,$$

so $D_\tau\psi(0,\tau_0)=0$. Consequently the third term is null. Recall now that $B_0 := D_h G(0,\alpha_0,\tau_0)$ and by Proposition 36, it holds that $QB_0=0$. So the second term is also null. Finally, using again that $D_v\psi(0,\alpha_0,\tau_0)\cdot\cos = 0$ we obtain the stated formula. $\square$

By Proposition 33, we have for all $h\in C^{0,0}_{2\pi}$

$$D_h G(0,\alpha,\tau)\cdot h = h - J(\alpha)\Theta_{\alpha,\tau} * h,$$

where the function $\Theta_{\alpha,\tau}$ is given by equation (59). It follows that

$$D^2_{h\tau}G(0,\alpha_0,\tau_0)\cdot 2\cos = -2J(\alpha_0)\frac{\partial}{\partial\tau}\left.(\Theta_{\alpha_0,\tau} * \cos)\right|_{\tau=\tau_0},$$

and so we have

$$D_\tau\tilde{\Phi}(0,\alpha_0,\tau_0) = -J(\alpha_0)\frac{\partial}{\partial\tau}\left.\widehat{\Theta}_{\alpha_0,\tau}(i)\right|_{\tau=\tau_0}.$$

Similarly,

$$D_\alpha\tilde{\Phi}(0,\alpha_0,\tau_0) = -\frac{\partial}{\partial\alpha}\left.\left(J(\alpha)\widehat{\Theta}_{\alpha,\tau_0}(i)\right)\right|_{\alpha=\alpha_0}.$$

Lemma 40. *Write $J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0}) =: x_0 + iy_0$. It holds that*

1. $D_\tau\tilde{\Phi}(0,\alpha_0,\tau_0) = (ix_0 - y_0)/\tau_0^2$.
2. $D_\alpha\tilde{\Phi}(0,\alpha_0,\tau_0) = \mathfrak{Z}_0'(\alpha_0)(x_0+iy_0)$, *where $\mathfrak{Z}_0'(\alpha_0)$ is defined in Lemma 11.*

Proof. From $\Theta_{\alpha,\tau}(t) = \tau\Theta_\alpha(\tau t)$, we have

$$\frac{\partial}{\partial\tau}\Theta_{\alpha,\tau}(t) = \tfrac{1}{\tau}\left[\tau\Theta_\alpha(\tau t) + \tau\Pi_\alpha(\tau t)\right], \quad \text{with} \quad \Pi_\alpha(t) := t\frac{\partial}{\partial t}\Theta_\alpha(t).$$

So

$$\left[\widehat{\frac{\partial}{\partial\tau}\Theta_{\alpha,\tau}}\right](z) = \tfrac{1}{\tau}\left[\widehat{\Theta}_\alpha(\tfrac{z}{\tau}) + \widehat{\Pi}_\alpha(\tfrac{z}{\tau})\right].$$

Moreover, an integration by parts shows that

$$\begin{aligned}
\widehat{\Pi}_\alpha(z) &= \int_0^\infty e^{-zt} t \frac{\partial}{\partial t} \Theta_\alpha(t) dt \\
&= -\widehat{\Theta}_\alpha(z) + z \int_0^\infty e^{-zt} t \Theta_\alpha(t) dt. \\
&= -\widehat{\Theta}_\alpha(z) - z \frac{\partial}{\partial z} \widehat{\Theta}_\alpha(z).
\end{aligned}$$

Choosing $z = i$ ends the proof of the first point. Define now

$$\Delta(z, \alpha) := J(\alpha) \widehat{\Theta}_\alpha(z) - 1.$$

By the definition of $\mathfrak{z}_0(\alpha)$ (see Lemma 11), we have

$$\forall \alpha \in V_{\alpha_0}, \quad \Delta(\mathfrak{z}_0(\alpha), \alpha) = 0.$$

We differentiate with respect to α and obtain

$$\frac{\partial}{\partial z} \Delta(\mathfrak{z}_0(\alpha), \alpha) \mathfrak{z}_0'(\alpha) + \frac{\partial}{\partial \alpha} \Delta(\mathfrak{z}_0(\alpha), \alpha) = 0.$$

Evaluating this expression at $\alpha = \alpha_0$ gives

$$\frac{\partial}{\partial \alpha} \left(J(\alpha) \widehat{\Theta}_\alpha \right) \Big|_{\alpha = \alpha_0} (\tfrac{i}{\tau_0}) = -\mathfrak{z}_0'(\alpha_0)(x_0 + i y_0),$$

which concludes the proof. $\square$

Lemma 41. *There exists $v_0 > 0$, $V_{\alpha_0} \times V_{\tau_0}$ an open neighborhood of (α_0, τ_0) in $(\mathbb{R}_+^*)^2$ and two functions $v \mapsto \tau_v, \alpha_v \in \mathcal{C}^1((-v_0, v_0))$ such that for all $(v, \alpha, \tau) \in (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0}$ we have*

$$\tilde{\Phi}(v, \alpha, \tau) = 0 \iff \tau = \tau_v \text{ and } \alpha = \alpha_v.$$

Proof. We decompose $\tilde{\Phi}$ into real part and imaginary part (without changing the notations), such that now

$$\tilde{\Phi} : (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0} \to \mathbb{R}^2.$$

We have $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$ and

$$\begin{aligned}
D_{(\alpha,\tau)} \tilde{\Phi}(0, \alpha_0, \tau_0) &= \begin{pmatrix} \Re D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Re D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \\ \Im D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Im D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \end{pmatrix} \\
&= \begin{pmatrix} x_0 \Re \mathfrak{z}_0'(\alpha_0) - y_0 \Im \mathfrak{z}_0'(\alpha_0) & -\frac{y_0}{\tau_0^2} \\ x_0 \Im \mathfrak{z}_0'(\alpha_0) + y_0 \Re \mathfrak{z}_0'(\alpha_0) & \frac{x_0}{\tau_0^2} \end{pmatrix}.
\end{aligned}$$

The determinant of this matrix is $\frac{\Re \mathfrak{z}_0'(\alpha_0)}{\tau_0^2}(x_0^2 + y_0^2)$ and this quantity is non-null by Assumptions 8 and 12. Consequently, the Implicit Function Theorems applies and gives the result. $\square$

The proof of Proposition 32 then follows immediately from this result and Lemma 38. This ends the proof of Theorem 14.

4 An explicit example

We now give a simple example of functions f and b such that Hopf bifurcations occurs and that the spectral assumptions of Theorem 14 can be analytically verified. Our minimal example satisfies all the assumptions of Theorem 14, except Assumption 3, because the function f we consider is not continuous. Indeed, to simplify the computation, we consider the step function

$$\forall x \in \mathbb{R}_+, \quad f(x) := \begin{cases} 0 & \text{for } 0 \le x < 1, \\ \beta^{-1} & \text{for } x \ge 1, \end{cases}$$

where $\beta > 0$ is a (small) parameter of the model.

4.1 Some generalities when f is a step function

We shall specify later the exact shape of b, for now we only assume that

$$\inf_{x\in[0,1]} b(x) > 0.$$

This ensures in particular that the Dirac mass at 0 is not an invariant measure. We now consider some fixed constant $\alpha \geq 0$. Let, for all $x \in [0,1]$

$$t^*_\alpha(x) := \inf\{t \geq 0,\ \varphi^\alpha_t(x) = 1\},$$

the time required for the deterministic flow to hit 1, starting from x. A simple computation show that

$$t^*_\alpha(x) = \int_x^1 \frac{dy}{b(y)+\alpha}.$$

Let $H^x_\alpha(t)$ be defined by (5) (with $\nu = \delta_x$, $\boldsymbol{a} \equiv \alpha$ and $s = 0$). Using the explicit shape of f, we find for all $x \in [0,1]$,

$$H^x_\alpha(t) := \begin{cases} 1 & \text{for } 0 \leq t < t^*_\alpha(x), \\ e^{-\frac{t-t^*_\alpha(x)}{\beta}} & \text{for } t \geq t^*_\alpha(x). \end{cases}$$

Moreover, for $x > 1$, it holds that $H^x_\alpha(t) = e^{-t/\beta}$. Altogether,

$$\forall z \in \mathbb{C} \text{ with } \Re(z) > -\beta^{-1}, \quad \widehat{H}_\alpha(z) = \frac{1-e^{-zt^*_\alpha(0)}}{z} + \frac{e^{-zt^*_\alpha(0)}}{z+\beta^{-1}}.$$

Note that in particular (using that $1/\gamma(\alpha) = \widehat{H}_\alpha(0)$)

$$1/\gamma(\alpha) = t^*_\alpha(0) + \beta.$$

So

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)} = \int_0^1 \frac{dy}{1+b(y)/\alpha} + \alpha\beta,$$

is a strictly increasing function of α: for a fixed value of $J > 0$, there is a unique $\alpha > 0$ solution of $\alpha = J\gamma(\alpha)$ and the corresponding ν^∞_α is the unique invariant measure of (1). Let $\sigma_\alpha = \lim_{t\to\infty} \varphi^\alpha_t(0)$. This invariant measure is given by

$$\nu^\infty_\alpha(x) = \begin{cases} \frac{\gamma(\alpha)}{b(x)+\alpha} & \text{for } x \in [0,1), \\ \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\frac{1}{\beta}\int_1^x \frac{dy}{b(y)+\alpha}\right) & \text{for } x \in [1,\sigma_\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $x \in [0,1]$ and $t > t^*_\alpha(x)$,

$$\frac{d}{dx} H^x_\alpha(t) = -\frac{1}{\beta}\frac{e^{-\frac{t-t^*_\alpha(x)}{\beta}}}{b(x)+\alpha}.$$

So the Laplace transform of $\frac{d}{dx}H^x_\alpha(t)$ is, for all $z \in \mathbb{C}$ with $\Re(z) > -\beta$

$$\forall x \in [0,1], \quad \int_0^\infty e^{-zt}\frac{d}{dx}H^x_\alpha(t)dt = -\frac{e^{-t^*_\alpha(x)z}}{b(x)+\alpha}\frac{1}{1+\beta z}.$$

Consequently for all $z \in \mathbb{C}$ with $\Re(z) > -\beta$

$$\begin{aligned} J(\alpha)\widehat{\Psi}_\alpha(z) &= -\frac{\alpha}{\gamma(\alpha)}\int_0^{\sigma_\alpha}\int_0^\infty e^{-zt}\frac{d}{dx}H^x_\alpha(t)dt\ \nu^\infty_\alpha(x)dx \\ &= \frac{\alpha}{1+\beta z}\int_0^1 \frac{e^{-t^*_\alpha(x)z}}{(b(x)+\alpha)^2}dx. \end{aligned}$$

Finally, the change of variable

$$x = \varphi_u^\alpha(0), \quad u \in [0, t_\alpha^*(0)),$$

such that $t_\alpha^*(x) = t_\alpha^*(0) - u$, shows that

$$J(\alpha)\widehat{\Psi}_\alpha(z) = \frac{\alpha e^{-zt_\alpha^*(0)}}{1+\beta z}\int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0))+\alpha}du.$$

So, the (local) stability of the invariant measure ν_α^∞ is given by the location of the roots of the following holomorphic function, defined for all $\Re(z) > -\beta^{-1}$:

$$\boxed{J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha e^{-zt_\alpha^*(0)}}{1+\beta z}\int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0))+\alpha}du - \frac{1-e^{-zt_\alpha^*(0)}}{z} - \frac{\beta e^{-zt_\alpha^*(0)}}{1+\beta z}}.$$

4.2 A linear drift b.

We now specify the shape of b. We choose:

$$\forall x \geq 0, \quad b(x) = m - x,$$

for some parameter $m > 1$, such that $b(x) + \alpha = \sigma_\alpha - x$, with $\sigma_\alpha = m + \alpha$. We then have $\varphi_u^\alpha(0) = \sigma_\alpha(1 - e^{-u})$ and so

$$t_\alpha^*(0) = \log\left(\frac{\sigma_\alpha}{\sigma_\alpha - 1}\right).$$

Finally

$$\int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0))+\alpha}du = \frac{1}{\sigma_\alpha}\int_0^{t_\alpha^*(0)} e^{(z+1)u}du = \frac{1}{\sigma_\alpha}\frac{e^{(z+1)t_\alpha^*(0)}-1}{z+1},$$

so

$$J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha}{\sigma_\alpha}\frac{e^{t_\alpha^*(0)} - e^{-zt_\alpha^*(0)}}{(1+\beta z)(z+1)} - \frac{1-e^{-zt_\alpha^*(0)}}{z} - \frac{\beta e^{-zt_\alpha^*(0)}}{1+\beta z}.$$

Consequently, we have to study the complex roots of the function

$$z \mapsto \frac{\alpha}{m+\alpha-1}\frac{1-\left(\frac{m+\alpha}{m+\alpha-1}\right)^{-(z+1)}}{(1+\beta z)(z+1)} - \frac{1-\left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{z} - \frac{\beta\left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{1+\beta z}.$$

Remark 42. *In fact this analysis can be easily extended to any linear drift*

$$b(x) = \kappa(m - x),$$

with $\kappa, m \in \mathbb{R}$. Indeed, adapting slightly the proof of [Cor20, Th. 21] when $\kappa \leq 0$, it holds that $f + b' \geq 0$ and so the unique non trivial invariant measure is locally stable: there is no Hopf bifurcation. If on the other hand $\kappa > 0$, by setting

$$\tilde{\kappa} = 1, \quad \tilde{\alpha} = \frac{\alpha}{\kappa}, \quad \tilde{m} = m \quad \tilde{\beta} = \kappa\beta,$$

we can easily reduce the problem to $\kappa = 1$.

We now make the following change of variable

$$\omega := \log\left(\frac{m+\alpha}{m+\alpha-1}\right) \quad \text{and} \quad \delta := \frac{\alpha}{m+\alpha-1},$$

with $\omega > 0$ et $\delta \in (0, 1)$. That is, we have

$$\alpha = \frac{\delta}{e^\omega - 1}, \quad m = 1 + \frac{1-\delta}{e^\omega - 1}.$$

With this change of variable, the equation becomes

$$\Re(z) > -\beta^{-1}, \quad \delta \frac{1}{1+\beta z}\frac{1-e^{-\omega(z+1)}}{1+z} - \frac{1-e^{-\omega z}}{z} - \frac{\beta e^{-\omega z}}{1+\beta z} = 0.$$

Assume now that

$$\beta + \omega - \delta(1-e^{-\omega}) \neq 0, \tag{66}$$

such that $z = 0$ is not a solution of the equation. Multiplying by $(1+\beta z)z$ on both side, we finally find that we have to study the roots of the following function

$$\Re(z) > -\beta^{-1}, \quad U(\beta, \delta, \omega, z) = 0,$$

with

$$\boxed{U(\beta, \delta, \omega, z) := \delta \frac{z}{z+1}(1-e^{-\omega(z+1)}) + e^{-\omega z} - (1+\beta z).}$$

4.3 On the roots of U

An explicit parametrization of the purely imaginary roots

We now describe all the imaginary roots of U. If $z = iy,\ y \geq 0$, the equation $U(\beta, \delta, \omega, z) = 0$ yields

$$\begin{cases} \cos(\omega y) + \sin(\omega y) y(1-\delta e^{-\omega}) &= 1 - \beta y^2 \\ -\sin(\omega y) + \cos(\omega y) y(1-\delta e^{-\omega}) &= y(1+\beta-\delta). \end{cases} \tag{67}$$

For $\omega > 0$ et $y \geq 0$ fixed, (67) admits a unique solution in (β, δ), given by

$$\beta_\omega^0(y) := \frac{(1+e^\omega)(1-\cos(\omega y)) - (e^\omega - 1)y\sin(\omega y)}{y^2 e^\omega - y^2\cos(\omega y) - y\sin(\omega y)},$$
$$\delta_\omega^0(y) := \frac{e^\omega(1+y^2)(1-\cos(\omega y))}{y^2 e^\omega - y^2\cos(\omega y) - y\sin(\omega y)}.$$

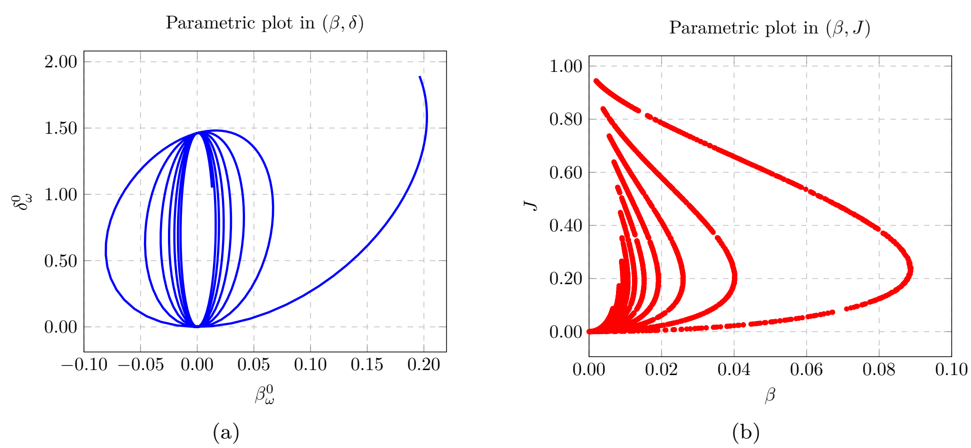


Figure 2: Description of the purely imaginary roots of U. (a) The parametric curve $(\beta_\omega^0(y), \delta_\omega^0(y))$, plotted with $\omega = 1$ et $y \in [0, 15.5\pi]$. Each point of the curve corresponds to a purely imaginary roots of U. (b) Purely imaginary solutions of U plotted in the plane (β, J), the value of m being fixed ($m = 3/2$).

Proposition 43. *The parametric curve* $(\beta_\omega^0(y), \delta_\omega^0(y))_{y>0}$ *admits exactly two multiple points given by*

$$(0,0) \quad \text{and} \quad (0, \frac{2}{1+e^{-\omega}}).$$

Apart from those two points, the curve does not intersect itself.

Proof. Squaring the two equations of (67) and summing the result, one gets

$$1 + y^2(1 - \delta e^{-\omega})^2 = (1 - \beta y^2)^2 + y^2(1 + \beta - \delta)^2,$$

that is

$$(1 - \delta e^{-\omega})^2 = -2\beta + \beta^2 y^2 + (1 + \beta - \delta)^2. \tag{68}$$

Note that if $\beta \neq 0$, for fixed values of δ, β, there is a unique y satisfying this equation. This proves that all the multiple points are located on the axis $\beta = 0$. When $\beta = 0$, the equation becomes

$$(1 - \delta e^{-\omega})^2 = (1 - \delta)^2,$$

whose solutions are

$$\delta = 0 \quad \text{and} \quad \delta = \frac{2}{1 + e^{-\omega}}.$$

Those are indeed multiple points. For $(0, 0)$ for instance, it suffices to consider $y = \frac{2\pi k}{\omega}, k \in \mathbb{N}^*$. This ends the proof. $\square$

4.4 Construction of the bifurcation point satisfying all the spectral assumptions.

Let $\omega_0 > 0$ being fixed, chosen arbitrarily. Let $y_0 := \frac{2\pi}{\omega_0}(1 - \frac{\epsilon_0}{\omega_0})$ with $\epsilon_0 > 0$ (small) to be chosen later. Let $\beta_0 := \beta^0_{\omega_0}(y_0)$ and $\delta_0 := \delta^0_{\omega_0}(y_0)$. We have

$$\beta_0 = \epsilon_0 + \mathcal{O}(\epsilon_0^2) \quad \text{as } \epsilon_0 \to 0.$$

and

$$\delta_0 = \frac{e^{\omega_0}}{2(e^{\omega_0} - 1)} \left(1 + \frac{(2\pi)^2}{\omega_0^2}\right) \epsilon_0^2 + \mathcal{O}(\epsilon_0^2) \quad \text{as } \epsilon_0 \to 0.$$

We then have

$$\frac{\partial U}{\partial z}(\beta_0, \delta_0, \omega_0, iy_0) = -\omega_0 - (1 + 2i\pi)\epsilon_0 + \mathcal{O}(\epsilon_0^2) \quad \text{as } \epsilon_0 \to 0.$$

This quantity is non-null provided that ϵ_0 is sufficiently small. The Implicit Function Theorem applies and gives the existence of a $\mathcal{C}^1$ function

$$(\beta, \delta, \omega) \mapsto z_0(\beta, \delta, \omega),$$

defined in the neighborhood of $(\beta_0, \delta_0, \omega_0)$ and such that

$$U(\beta, \delta, \omega, z_0(\beta, \delta, \omega)) = 0, \quad \text{with} \quad z_0(\beta_0, \delta_0, \omega_0) = iy_0.$$

Furthermore, one has

$$\frac{\partial}{\partial \delta} z_0(\beta_0, \delta_0, \omega_0) = -\frac{\frac{\partial U}{\partial \delta}(\beta_0, \delta_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, \delta_0, \omega_0, iy_0)} = 2\pi \frac{1 - e^{-\omega_0}}{\omega_0} \frac{2\pi + i\omega_0}{(2\pi)^2 + \omega_0^2} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0$$

and

$$\frac{\partial}{\partial \omega} z_0(\beta_0, \delta_0, \omega_0) = -\frac{\frac{\partial U}{\partial \omega}(\beta_0, \delta_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, \delta_0, \omega_0, iy_0)} = -\frac{2i\pi}{\omega_0} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0.$$

We finally set

$$\alpha_0 := \frac{\delta_0}{e^{\omega_0} - 1}, \quad m_0 := 1 + \frac{1 - \delta_0}{e^{\omega_0} - 1},$$

and

$$\mathfrak{z}_0(\alpha) := z_0(\beta_0, \frac{\alpha}{m_0 + \alpha - 1}, \log\left(\frac{m_0 + \alpha}{m_0 + \alpha - 1}\right)),$$

such that

$$\begin{aligned}\frac{d}{d\alpha}\mathfrak{z}_0(\alpha_0) =& 2\pi\frac{1-e^{-\omega_0}}{\omega_0}\frac{2\pi+i\omega_0}{(2\pi)^2+\omega_0^2}\frac{m_0-1}{(m_0-1-\alpha_0)^2}\\ &+\frac{2i\pi}{\omega_0}\frac{1}{(m_0-1+\alpha_0)(m_0+\alpha_0)}+\mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0\to 0.\end{aligned}$$

The second term on the right hand side is purely imaginary. So

$$\Re\frac{d}{d\alpha}\mathfrak{z}_0(\alpha_0) = \frac{1-e^{-\omega_0}}{\omega_0}\frac{(2\pi)^2}{(2\pi)^2+\omega_0^2}\frac{m_0-1}{(m_0-1-\alpha_0)^2}+\mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0\to 0.$$

This quantity is strictly positive provided that ϵ_0 is small enough. By choosing the parameters of the model to be $\beta=\beta_0$ and $m=m_0$, the Assumptions 8, 9 and 12 are satisfied at the point $\alpha=\alpha_0$.

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